

Practice Midterm II Examination
Time Limit: 50 Minutes

November 25 2024

This examination document contains 5 pages, including this cover page, and 4 problems. You must verify whether there are any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- (A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.
- (B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.
- (C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- (D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) Solve the following parts:

(a) (10 points) Let $X = \left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\}$. Show that $\sup(X) = 2$.

(b) (10 points) Consider the sequence $x_n = \frac{2n+1}{n+1}$, $n \in \mathbb{N}$. Show that x_n is increasing.

(c) (5 points) Prove that the sequence (x_n) converges.

2. (25 points) Solve the following two parts:

(a) (15 points) Consider the sequence $(x_n)_n \in \mathbb{N}$ given by

$$x_n = \frac{4n! + 2^n}{n^n}.$$

Show that $\lim_{n \rightarrow \infty} x_n = 0$.

(b) (15 points) Prove that the sequence $(y_n)_n \in \mathbb{N}$ given by

$$y_n = 2^n \left(1 - \frac{1}{n^3}\right)$$

does *not* converge.

3. (25 points) Solve the following two parts:

(a) (15 points) Prove that $\sqrt[7]{5} \in \mathbb{R}$ is not a rational number.

(b) (10 points) Give an example of a sequence $(x_n)_n$ of rational numbers, $x_n \in \mathbb{Q}$, that converges to an irrational number.

4. (25 points) Solve the following two problems:

(a) (15 points) Consider the map $f : \mathbb{Q} \rightarrow \mathbb{Q}$ given by $f(x) = 5x - 8$. Show that f is a bijection.

(b) (10 points) Show that the set

$$X = \{p(x) : p(x) = a_0 + a_1x + \dots + a_{23}x^{23} + a_{24}x^{24}, \quad a_1, a_2, \dots, a_{23}, a_{24} \in \mathbb{Z}\}$$

of polynomials of degree 24 with integer coefficients is countable.