University of California Davis **Name (Print):** ______________

Differential Equations MAT 108 **Student ID (Print):** ______________

Practice Midterm II Exam 2 **November 25 2024**

Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Solve the following parts:

   (a) (10 points) Show that the set

   $$X = \left\{ \frac{1}{3n^4 + 7n^2 + 5} : n \in \mathbb{N}, \quad n \equiv 4 \pmod 5 \right\}$$

   is bounded (i.e. it is bounded above and bounded below).

   (b) (10 points) Let $Y = [2, 5) \cap \mathbb{Q} = \{y \in \mathbb{R} : 2 \leq y < 5, y \in \mathbb{Q}\}$.
   Find $\sup(Y)$ and $\inf(Y)$ or show they do not exist. (Please prove your answer.)

2. (25 points) Solve the following parts:

   (a) (10 points) Consider the recursive sequence (x_n), $n \in \mathbb{N}$, given by

   $$x_{n+1} = \sqrt{3x_n + 10}, \quad x_1 = 1.$$

   Show that $(x_n)_n$ is bounded above by 5.

   (b) (5 points) Prove that (x_n) is increasing.

(c) (5 points) Show that (x_n) converges.

(d) (5 points) Find the limit $\lim_{n\to\infty} x_n$.

3. (25 points) Consider the map

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^3 - x.$$

(a) (15 points) Prove that f is *not* injective.

(b) (10 points) Prove that f is surjective.

4. (25 points) Solve the following two problems:

   (a) (15 points) Let $m \in \mathbb{N}$ be a fixed natural number and the set $[m] = \{1, 2, \ldots, m\}$. Consider the set

   $$X = \{f : [m] \longrightarrow \mathbb{N}\}$$

   of maps from $[m]$ to $\mathbb{N}$. Show that X is countable.

   (b) (10 points) Show that the set $T = \{(x_n) : n \in \mathbb{N}, x_n \in \{-1, 0, 1\}\}$ of ternary sequences is uncountable.[1]

[1] That is, an element of T is a sequence (x_n), indexed by $n \in \mathbb{N}$, where each value x_n can be $-1, 0$ or 1.