University of California Davis **Name (Print):** ______________

Differential Equations MAT 108 **Student ID (Print):** ______________

Practice Midterm II Exam 2 **November 25 2024**

Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Solve the following parts:

(a) (10 points) Show that the set

$$X = \left\{ \frac{1}{3n^4 + 7n^2 + 5} : n \in \mathbb{N}, \quad n \equiv 4 \pmod 5 \right\}$$

is bounded (i.e. it is bounded above and bounded below).

Proof. Consider $n \in \mathbb{N}$ where $n \equiv 4 \pmod 5$. Then n is always positive. As such, $\frac{1}{3n^4+7n^2+5}$ is always positive. Thus

$$\frac{1}{3n^4 + 7n^2 + 5} \geq 0$$

and X is bounded below by 0.
Since $n \geq 0$, we have $3n^4 + 7n^2 + 5 \geq 5$. Thus

$$\frac{1}{3n^4 + 7n^2 + 5} \leq \frac{1}{5}.$$

Thus X is bounded above by $\frac{1}{5}$. □

(b) (10 points) Let $Y = [2, 5) \cap \mathbb{Q} = \{y \in \mathbb{R} : 2 \leq y < 5, y \in \mathbb{Q}\}$.
Find $\sup(Y)$ and $\inf(Y)$ or show they do not exist. (Please prove your answer.)

Proof. We will show that the $\inf(Y) = 2$. First, observe 2 is a lower bound since for all $y \in Y$, $y \in [2, 5)$ and as such $2 \leq y$. For the sake, of contradiction, assume there exists some lower bound ℓ such that $2 < \ell$. Note $\ell < 3$ since $3 \in Y$. Since the rationals are dense in the reals, there exists some $x \in \mathbb{Q}$ such that $2 < x < \ell < 3$. Thus $x \in \mathbb{Q}$ and $x \in [2, 5)$. That is, $x \in Y$ where $x < \ell$. This contradicts that ℓ is a lower bound. Therefore, 2 must be our greatest lower bound.
We will show also that $\sup(Y) = 5$. First, observe 5 is an upper bound since for all $y \in Y$, $y \in [2, 5)$ and as such $y < 5$. For the sake of contradiction, assume there is some upper bound u such that $u < 5$. Note $4 < u$ as $4 \in Y$. Since the rationals are dense in the reals there exists some rational number q such that $4 < u < q < 5$. So $q \in [2, 5)$ and $q \in \mathbb{Q}$. Thus $q \in Y$ with $u < q$. This contradicts that q is an upper bound. Therefore, $\sup(Y) = 5$. □

**Note: be careful using midpoint here as u, ℓ are not guaranteed to be rational. So the midpoint may not be in the set!

2. (25 points) Solve the following parts:

(a) (10 points) Consider the recursive sequence (x_n), $n \in \mathbb{N}$, given by

$$x_{n+1} = \sqrt{3x_n + 10}, \quad x_1 = 1.$$

Show that $(x_n)_n$ is bounded above by 5.

Proof. We will use induction to prove this.
Base case: Observe $1 \leq 5$. As such $x_1 \leq 5$ and our base case holds.
Inductive Step: Assume $x_n \leq 5$ for some $n \in \mathbb{N}$. Observe $n + 1 \geq 2$. As such,

$$\begin{aligned} x_{n+1} &= \sqrt{3x_n + 10} \\ &\leq \sqrt{3(5) + 10} && \text{by Inductive Hypothesis} \\ &= \sqrt{25} \\ &= 5. \end{aligned}$$

Note the above inequality works since $\sqrt{}$ respects inequalities. Thus $x_{n+1} \leq 5$.
Therefore, by induction, $x_n \leq 5$ for all $n \in \mathbb{N}$. □

(b) (5 points) Prove that (x_n) is increasing.

Proof. We will use induction to prove this.
Base case: Observe $x_2 = \sqrt{3 + 10} = \sqrt{13} \geq \sqrt{1} = 1 = x_1$. Thus $x_1 \leq x_2$ and the base case holds.
Inductive Step: Assume for some $n \in \mathbb{N}$, we have $x_n \leq x_{n+1}$. WWTS $x_{n+1} \leq x_{n+2}$. Observe

$$\begin{aligned} x_{n+2} &= \sqrt{3x_{n+1} + 10} \\ &\geq \sqrt{3x_n + 10} && \text{by the inductive hypothesis} \\ &= x_{n+1}. \end{aligned}$$

Thus $x_{n+1} \leq x_{n+2}$.
Therefore, by induction, $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. As such, (x_n) is increasing. □

(c) (5 points) Show that (x_n) converges.

Proof. By part a, we know (x_n) is bounded above by 5. By part b, we know (x_n) is increasing. Therefore, by the monotone convergence theorem, (x_n) converges. □

(d) (5 points) Find the limit $\lim_{n\to\infty} x_n$.

Proof. We know by part c that the limit exists. Call it

$$\lim_{n\to\infty} x_n = x.$$

Since $x_n \geq 0$ for all $n \in \mathbb{N}$, $x \geq 0$ Since the limit exists, $\lim_{n\to\infty} 3x_n + 10$ exists. Observe, because the limits exits, we can do the following.

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} \sqrt{3x_n + 10} \\ &= \sqrt{\lim_{n\to\infty} (3x_n + 10)} \\ &= \sqrt{3 \lim_{n\to\infty} x_n + 10} \\ &= \sqrt{3x + 10}. \end{aligned}$$

So $x^2 = |3x + 10| = 3x + 10$ since $x \geq 0$. Thus $0 = x^2 - 3x - 10 = (x - 5)(x + 2)$. Thus x must be either 5 or -2. Since $x \geq 0$, $x \neq -2$. Therefore,

$$\lim_{n\to\infty} x_n = 5.$$

□

3. (25 points) Consider the map

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^3 - x.$$

(a) (15 points) Prove that f is *not* injective.

Proof. Consider $-1, 1 \in \mathbb{R}$. Observe $f(-1) = (-1)^3 - (-1) = -1 + 1 = 0$ and $f(1) = 1^3 - 1 = 0$. Thus $f(-1) = f(1)$ but $-1 \neq 1$. As such, f is not injective. □

(b) (10 points) Prove that f is surjective.

Proof. First observe that f is a continuous function. So if $f(x) < f(y)$ for some $x, y \in \mathbb{R}$, then by the intermediate value theorem all values between $f(x)$ and $f(y)$ must be obtained. We will know show that $\{f(x) : x \in \mathbb{R}\}$ is not bounded above and not bounded below.

Consider some $M \in \mathbb{N}$. We will show that we can always find an x such that $f(x) > M$. If $M = 1$ then $f(2) = 2^3 - 2 = 6 > 1$. Now consider when $M \geq 2$. Then

$$f(M) = M^3 - M \geq 2^2 M - M \geq 3M > M.$$

Thus $\{f(x) : x \in \mathbb{R}\}$ is not bounded above.

Now consider $-N$ where $N \in \mathbb{N}$. We will show that we can always find an x such that $f(x) < -N$. If $-N = -1$ then $f(-2) = -8 + 2 = -6 < -1$. Now consider when $-N \leq -2$. Then

$$f(-N) = -N^3 + N = -(N^3 - N) = -f(N)$$

where $N \geq 2$. Thus by the above $f(N) > N$ as $N \geq 2$. Thus $f(-N) = -f(N) < -N$. Thus $\{f(x) : x \in \mathbb{R}\}$ is not bounded below.

Recall f is continuous and $f(0) = 0$. Since f is not bounded above, by the intermediate value theorem every positive number can be obtained for $f(x)$. Likewise, since f is not bounded below, by the intermediate value theorem, every negative number can be obtained for $f(x)$. As such, f is surjective. □

4. (25 points) Solve the following two problems:

(a) (15 points) Let $m \in \mathbb{N}$ be a fixed natural number and the set $[m] = \{1, 2, \dots, m\}$. Consider the set

$$X = \{f : [m] \longrightarrow \mathbb{N}\}$$

of maps from $[m]$ to $\mathbb{N}$. Show that X is countable.

Proof. We will recursively define a collection of sets. Consider the set

$$X_{(i)} = \{f : [m] \longrightarrow \mathbb{N} : f(1) = i\}$$

where $i \in \mathbb{N}$. Note if $i_1 \neq i_2$ then $X_{i_1} \cap X_{i_2} = \emptyset$. Then

$$X = \bigcup_{i=1}^{\infty} X_i.$$

Define

$$X_{(i_1,i_2,\dots,i_k)} = \{f \in X_{(i_1,i_2,\dots,i_{k-1})} : f(k) = i_k\}$$

where $k \in \{2, ..., m\}$ and $i_n \in \mathbb{N}$ for all n. Thus we have

$$X_{(i_1,i_2,\dots,i_{k-1})} = \bigcup_{i_k=1}^{\infty} X_{(i_1,i_2,\dots,i_k)}$$

where this is a union over pairwise disjoint sets. Therefore,

$$X = \bigcup_{i_1=1}^{\infty} \bigcup_{i_2=1}^{\infty} \bigcup_{i_3=1}^{\infty} \cdots \bigcup_{i_m=1}^{\infty} X_{(i_1,i_2,\dots,i_m)} \tag{1}$$

Observe

$$X_{(i_1,i_2,\dots,i_m)} = \{f : [m] \longrightarrow \mathbb{N} : f(1) = i_1, f(2) = i_2, \cdots, f(m) = i_m\}.$$

Thus $X_{(i_1,i_2,\dots,i_m)}$ only has one element and is a finite set. Thus it is a countable set. Therefore, by (1), X is the countable union of countable sets. As such X is countable. □

(b) (10 points) Show that the set $T = \{(x_n) : n \in \mathbb{N}, x_n \in \{-1, 0, 1\}\}$ of ternary sequences is uncountable.[1]

[1] That is, an element of T is a sequence (x_n), indexed by $n \in \mathbb{N}$, where each value x_n can be $-1, 0$ or 1.

Proof. We consider a subset of of T defined as

$$S := \{(x_n) : n \in \mathbb{N}, x_n \in \{-0, 1\}\} \subset T.$$

We now define a function $f : S \to P(\mathbb{N})$ where $f((x_n)) = \{m \in \mathbb{N} : x_m = 1\}$. Consider the sequences (a_n), $(b_n) \in S$ when $f((a_n)) = f((b_n))$. That is,

$$\{m \in \mathbb{N} : a_m = 1\} = \{m \in \mathbb{N} : b_m = 1\}.$$

So the sequence (a_n) and (b_n) must be 1 on the same indices. However, everywhere else, they must both be zero. Thus $(a_n) = (b_n)$ as sequences. Therefore, f is injective. Now consider $A \in P(\mathbb{N})$. That is, $A \subseteq \mathbb{N}$. Define the sequence (a_n) as

$$a_n = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \end{cases}.$$

Then $f((a_n)) = \{m \in \mathbb{N} : a_m = 1\} = \{m \in \mathbb{N} : m \in A\} = A$. Thus f is surjective. Therefore, f is bijective.

Since f is bijective, the cardinalates of S and $P(A)$ must be the same. We know $\text{card}\mathbb{N} < \text{card}P(A)$. Therefore, $P(A)$ must be uncountable and as such so to must S be. However, S is a subset of T. So T has an uncountable subset. We know all subsets of countable sets are countable. So T cannot be countable. Therefore, T is uncountable. □