University of California Davis **Name (Print):** ____________
Differential Equations MAT 108 **Student ID (Print):** ____________

Practice Midterm II Examination

November 25 2024

Time Limit: 50 Minutes

This examination document contains 9 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Solve the following parts:

(a) (10 points) Let $X = \left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\}$. Show that $\sup(X) = 2$.

Let $\frac{2n+1}{n+1} \in X$, where $n \in \mathbb{N}$. Then,

$$\begin{aligned} \frac{2n+1}{n+1} &= \frac{2n+2}{n+1} - \frac{1}{n+1} \\ &= 2 - \frac{1}{n+1} \end{aligned}$$

Since $n \in \mathbb{N}$, $n > 0$, so $\frac{1}{n+1} > 0 \Rightarrow 2 - \frac{1}{n+1} < 2 - 0 = 2$. Thus, 2 is an upper bound of X.

Suppose for contradiction that $\sup(X) \neq 2$. Since 2 is an upper bound, $\sup(X) \leq 2$. Thus, $\sup(X) < 2$. So, $0 < 2 - \sup(X)$. By Proposition 10.4, there exists $N \in \mathbb{N}$ st. $\frac{1}{N} < 2 - \sup(X)$. Since $N < N+1$, $\frac{1}{N} > \frac{1}{N+1}$. Thus,

$$\frac{1}{N+1} < \frac{1}{N} < 2 - \sup(X) \Rightarrow \sup(X) < 2 - \frac{1}{N+1} = \frac{2N+1}{N+1}$$

Since $N \in \mathbb{N}$, $\frac{2N+1}{N+1} \in X$. This contradicts that the supremum of X must be greater than any element of X. So, $\sup(X) = 2$.

(b) (10 points) Consider the sequence $x_n = \frac{2n+1}{n+1}$, $n \in \mathbb{N}$. Show that x_n is increasing.
Let x_n be the terms of the sequence as defined above. Then,

$$\begin{aligned} x_{n+1} - x_n &= \frac{2(n+1)+1}{(n+1)+1} - \frac{2n+1}{n+1} \\ &= \frac{2n+3}{n+2} - \frac{2n+1}{n+1} \\ &= \frac{(2n+3)(n+1) - (2n+1)(n+2)}{(n+2)(n+1)} \\ &= \frac{2n^2+5n+3-2n^2-5n-2}{n^2+3n+1} \\ &= \frac{1}{n^2+3n+1} \end{aligned}$$

Since $n > 0$, $n^2+3n+1 > 0+0+1 = 1 > 0$. Thus, $x_{n+1} - x_n = \frac{1}{n^2+3n+1} > 0$. Thus, $x_{n+1} > x_n$ so the sequence is increasing.

(c) (5 points) Prove that the sequence (x_n) converges.
As shown in part (a), the sequence is bounded above by 2. As shown in part (b), the sequence is increasing. So by the Monotone Convergence theorem, the sequence (x_n) converges.

2. (25 points) Solve the following two parts:

   (a) (15 points) Consider the sequence $(x_n)_n \in \mathbb{N}$ given by

$$x_n = \frac{4n! + 2^n}{n^n}.$$

Show that $\lim_{n\to\infty} x_n = 0$.

Scratch work: For a given $\epsilon > 0$, we want to find a $N \in \mathbb{N}$ st. $n \geq N$ implies $|x_n - 0| < \epsilon$.

$$\begin{aligned}
|x_n - 0| &= \left|\frac{4n! + 2^n}{n^n}\right| \\
&= \frac{4n! + 2^n}{n^n} \\
&= \frac{4n!}{n^n} + \left(\frac{2}{n}\right)^n \\
&= 4\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^n \\
&\leq 4(1)(1)\cdots\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^n \\
&\leq 4(1)(1)\cdots\left(\frac{1}{n}\right) + \frac{2}{n} \\
&= \frac{6}{n}
\end{aligned}$$

when $n \geq 2$. Then,

$$\frac{6}{n} < \epsilon \Rightarrow n > \frac{6}{\epsilon}$$

Proof: Let $\epsilon > 0$ be given. Then, by proposition 10.4, there exist $N' \in \mathbb{N}$ st. $N' > \frac{6}{\epsilon}$. Let $N \in \mathbb{N}$ be st. $N = \max\{N', 2\}$, ie. $N \geq N'$ and $N \geq 2$. Suppose $n \in \mathbb{N}$ st.

$n \geq N$. Then,

$$\begin{aligned}
|x_n - 0| &= \left|\frac{4n! + 2^n}{n^n}\right| \\
&= \frac{4n! + 2^n}{n^n} && \text{since } n > 0 \\
&= \frac{4n!}{n^n} + \left(\frac{2}{n}\right)^n \\
&= 4\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\ldots\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^n \\
&\leq 4(1)(1)\ldots\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^n \\
&\leq 4(1)(1)\ldots\left(\frac{1}{n}\right) + \frac{2}{n} && \text{since } n \geq N \geq 2 \\
&= \frac{6}{n} \\
&\leq \frac{6}{N} && \text{since } N \geq N' > 6/\epsilon \\
&< \frac{6}{6/\epsilon} \\
&= \epsilon.
\end{aligned}$$

(b) (15 points) Prove that the sequence $(y_n)_n \in \mathbb{N}$ given by

$$y_n = 2^n \left(1 - \frac{1}{n^3}\right)$$

does *not* converge.

First, let's show that the sequence is unbounded above. Consider $n \in \mathbb{N}$ st. $n \geq 2$. Then,

$$\begin{aligned}
2^n \left(1 - \frac{1}{n^3}\right) &\geq 2^n \left(1 - \frac{1}{2^3}\right) && \text{since } n \geq 2 \\
&= 2^n \left(\frac{7}{2^3}\right) \\
&= 2^{n-3} \cdot 7 \\
&\geq 2^{n-3} \\
&\geq n - 3 && \text{since as shown on the midterm that } 2^n \geq n.
\end{aligned}$$

Since the set $\{n - 3 \mid n \in \mathbb{N}, n \geq 2\}$ contains $\mathbb{N}$, and $\mathbb{N}$ is not bounded above, neither can the set. So, for $n \geq 2$, $n - 3$ has no upper bound. Thus, $2^n \left(1 - \frac{1}{n^3}\right)$ cannot be bounded above for $n \geq 2$. Thus, the whole sequence cannot be bounded above.

Next, let's show that the sequence is increasing. Let $n \in \mathbb{N}$. Then,

$$\begin{aligned}
y_{n+1} - y_n &= 2^{n+1} \left(1 - \frac{1}{(n+1)^3}\right) - 2^n \left(1 - \frac{1}{n^3}\right) \\
&= 2^n \left(2 \left(1 - \frac{1}{(n+1)^3}\right) - \left(1 - \frac{1}{n^3}\right)\right) \\
&= 2^n \left(2 - \frac{2}{(n+1)^3} - 1 + \frac{1}{n^3}\right) \\
&= 2^n \left(1 + \frac{(n+1)^3 - 2n^3}{n^3(n+1)^3}\right) \\
&= 2^n \left(1 + \frac{n^3 + 3n^2 + 3n + 1 - 2n^3}{n^3(n+1)^3}\right) \\
&= 2^n \left(1 + \frac{-n^3 + 3n^2 + 3n + 1}{n^3(n+1)^3}\right) \\
&> 2^n \left(1 + \frac{-n^3}{n^3(n+1)^3}\right) && \text{since } 3n^2 + 3n + 1 > 0 \\
&= 2^n \left(1 - \frac{1}{(n+1)^3}\right) \\
&> 2^n && \text{since } \frac{1}{n+1} > 0.
\end{aligned}$$

Since $2^n > 0$ for all $n \in \mathbb{N}$, $y_{n+1} - y_n > 0 \Rightarrow y_{n+1} > y_n$. So, the sequence is increasing.
By Proposition 10.21, if an increasing sequence converges, the sequence is bounded above by the limit. Thus, since (y_n) is unbounded above and increasing, it cannot converge.

3. (25 points) Solve the following two parts:

   (a) (15 points) Prove that $\sqrt[7]{5} \in \mathbb{R}$ is not a rational number.

   Suppose for contradiction that $\sqrt[7]{5}$ is rational. Then there exists $m, n \in \mathbb{Z}$ st. m, n do not share any common factors and

$$\sqrt[7]{5} = \frac{m}{n}$$
$$\Rightarrow m^7 = 5n^7.$$

   Thus, 5 divides m^7. Since prime factorization is unique, every prime factor of m^7 must also be a prime factor of m. So, 5 divides m. Let $m = 5k$ for some $k \in \mathbb{Z}$. Then,

$$(5k)^7 = 5n^7$$
$$5^6 k^7 = n^7.$$

   So, 5 divides n^7. As shown previously, this means that 5 divides n. This contradicts that m, n do not share any common factors. So, $\sqrt[7]{5}$ is irrational.

   (b) (10 points) Give an example of a sequence $(x_n)_n$ of rational numbers, $x_n \in \mathbb{Q}$, that converges to an irrational number.

   Consider the sequence $(x_n)_n$ given by the recurrence relation:

$$x_0 = 2, x_{n+1} = x_n - \frac{x_n^7 - 5}{7x_n^6}$$

   This is a sequence of rational numbers. It is also the recurrence relation given by Newton's method for finding roots on the function $f(x) = x^7 - 5$. Thus, the sequence converges to $\sqrt[7]{5}$, which we showed is irrational in part a.

4. (25 points) Solve the following two problems:

   (a) (15 points) Consider the map $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ given by $f(x) = 5x - 8$.
   Show that f is a bijection.
   We can show that f is a bijection by demonstrating an inverse for f. If $f(x) = 5x - 8$, we can "solve"

$$\begin{aligned} 5x &= f(x) + 8 \\ x &= \frac{1}{5} f(x) + \frac{8}{5}. \end{aligned}$$

Now define a function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \frac{1}{5}x + \frac{8}{5}$. This is a well defined function on $\mathbb{Q}$, because $\frac{8}{5} \in \mathbb{Q}$, $\frac{1}{5} \in \mathbb{Q}$, and $\mathbb{Q}$ is closed under multiplication and addition. We now wish to show that the compositions $f \circ g$ and $g \circ f$ are both the identity function on $\mathbb{R}$. We have:

$$\begin{aligned} f \circ g(x) &= f\left(\frac{1}{5}x + \frac{8}{5}\right) \\ &= 5\left(\frac{1}{5}x + \frac{8}{5}\right) - 8 \\ &= x + 8 - 8 \\ &= x. \end{aligned}$$

Similarly,

$$\begin{aligned} g \circ f(x) &= g\left(5x - 8\right) \\ &= \frac{1}{5}\left(5x - 8\right) + \frac{8}{5} \\ &= x - \frac{8}{5} + \frac{8}{5} \\ &= x. \end{aligned}$$

Indeed, we see that f and g are two sided inverses of each other, so f is a bijection.

(b) (10 points) Show that the set

$$X = \{p(x) : p(x) = a_0 + a_1 x + \ldots + a_{23}x^{23} + a_{24}x^{24}, \quad a_1, a_2, \ldots, a_{23}, a_{24} \in \mathbb{Z}\}$$

of polynomials of degree 24 with integer coefficients is countable.

We can show by induction that the set X_n of polynomials of degree n with integer coefficients are always a countable set. The polynomials of degree zero are precisely the integers, and we know that $\mathbb{Z}$ is countable, so the base case follows.

Now suppose that for some $n \in \mathbb{N}$, X_n is countable. We can consider a countably infinite collection of countable sets, defined for $k \in \mathbb{Z}$ by

$$A_k = \{kx^{n+1} + p(x) \mid p(x) \in X_n\}.$$

Each A_k is the set of elements of X_{n+1} whose coefficient of x^{n+1} is k. In particular, we see that

$$X_{n+1} = \bigcup_{k \in \mathbb{Z}} A_k.$$

By proposition 13.19, the union of a countable sets is countable. By induction, we then see that X_n is countable for all n. In particular, this is true for $X = X_{24}$, so X is countable.