MAT 108: SOME SOLUTIONS TO PRACTICE PROBLEMS 2ND MIDTERM

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Abstract. This document contains solutions to some practice problems for the second part of the course, with a view towards the 2nd midterm.

Problem 1. Prove the following four statements.

(a) Let $X_1 = \{x \in \mathbb{R} : -2 \leq x < 3\} = [-2, 3)$. Then $\inf(X_1) = -2$ and $\sup(X_1) = 3$,

Solution (a). By contradiction, suppose that the infimum is $-2 < \inf(X_1)$, so that -2 is not the greatest lower bound. Then $\inf(X_1)$ cannot be a lower bound for X_1 since $-2 \in X_1$.

Let us show that $\sup(X_1) = 3$ now. By contradicition, suppose that the supremum is $\sup(X_1) < 3$, in order for 3 not to be the least upper bound. Since $\sup(X_1)$ is an upper bound, we must have

$$x < \sup(X_1), \quad \forall x \in X_1.$$

Consider $\delta = (3 + \sup(X_1))/2 \in \mathbb{R}^+$. Then we have

$$\delta = (3 + \sup(X_1))/2 < 3, \quad \text{since } \sup(X_1) < 3,$$

and so $\delta \in X_1$. In contrast, we also have

$$\delta = (3 + \sup(X_1))/2 > \sup(X_1), \quad \text{since } 3 > \sup(X_1),$$

and thus $\sup(X_1) < \delta$ and $\sup(X_1)$ is not an upper bound for X_1, a contradiction. Hence, $\sup(X_1) = 3$.

(b) Let $X_2 = \{x \in \mathbb{Q} : -\sqrt{3} \leq x < \sqrt{2}\} = [-\sqrt{3}, \sqrt{2}) \cap \mathbb{Q}$.
Then $\inf(X_2) = -\sqrt{3}$ and $\sup(X_2) = \sqrt{2}$,

(c) Let $X_3 = \{2^{-n} : n \in \mathbb{N}\}$, then $\inf(X_3) = 0$ and $\sup(X_3) = 1/2$,

(d) Let $X_4 = \left\{\dfrac{n}{5n+3} : n \in \mathbb{N}\right\}$, then $\inf(X_4) = 1/8$ and $\sup(X_4) = 1/5$.

Problem 2. Compute the following limits using the ε-definition of the limit of a sequence.

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^3} = 0$,

Solution (a). We want that $\forall \varepsilon > 0$ we have the inequality

$$\left|\frac{1}{n^3}\right| < \varepsilon, \quad \text{for } n \gg 1.$$

Indeed, given any $\varepsilon > 0$, by Proposition 10.4, we can choose $n \in \mathbb{N}$ such that

$$\left|\frac{1}{n}\right| < \varepsilon, \quad \text{for } n \gg 1.$$

Then, we will have

$$\left|\frac{1}{n^3}\right| < \left|\frac{1}{n}\right| \varepsilon, \quad \text{for } n \gg 1,$$

as required.

(b) $\lim_{n\to\infty} \frac{2^n}{7^n} = 0$,

(c) $\lim_{n\to\infty} \frac{(-1)^n \cdot 4^n}{n!} = 0$,

Solution (c). We want that $\forall \varepsilon > 0$ we have the inequality

$$\left| \frac{(-1)^n \cdot 4^n}{n!} \right| < \varepsilon, \quad \text{for } n \gg 1.$$

Indeed, given any $\varepsilon > 0$, by Proposition 10.4, we can choose $n \in \mathbb{N}$ such that

$$\left| \frac{1}{n} \right| < \frac{6\varepsilon}{4^4}, \quad \text{for } n \gg 1.$$

Then, we will have

$$\left| \frac{(-1)^n \cdot 4^n}{n!} \right| < \left| \frac{4^4}{6n} \right| < \varepsilon, \quad \text{for } n \gg 1,$$

as required.

(d) $\lim_{n\to\infty} \sqrt[n]{n} = 1$,

Solution (d). We want that $\forall \varepsilon > 0$ we have the inequality

$$\left| \sqrt[n]{n} - 1 \right| < \varepsilon, \quad \text{for } n \gg 1.$$

By the Binomial Theorem, we have that

$$\left| \sqrt[n]{n} - 1 \right| < \sqrt{\frac{2}{n-1}}.$$

Thus, given the $\varepsilon > 0$ above, we apply Proposition 10.4 to the positive quantity $\varepsilon^2/(\varepsilon^2+2)$, and we choose $n \in \mathbb{N}$ such that

$$\left| \frac{1}{n} \right| < \varepsilon^2/(\varepsilon^2+2), \quad \text{for } n \gg 1,$$

which is just saying

$$\left| \sqrt{\frac{2}{n-1}} \right| < \varepsilon, \quad \text{for } n \gg 1,$$

Then, we will have

$$\left| \sqrt[n]{n} - 1 \right| < \left| \sqrt{\frac{2}{n-1}} \right| < \varepsilon, \quad \text{for } n \gg 1,$$

as required.

(e) $\lim_{n\to\infty} \frac{3n+7}{8n+1} = \frac{3}{8}$,

(f) $\lim_{n\to\infty} \frac{5n^2+3n+7}{9n^2+17n} = \frac{5}{9}$,

Problem 3. For each of the following sequences, first decide whether the following sequences are eventually **increasing** or **decreasing** (or neither), and second decide whether they are **bounded above** or **below** (or neither). Use this to decide whether each of the sequences (x_n) converges.

(a) $x_n = \frac{1}{4n}$,

(b) $x_n = \frac{n^2(-1)^{n+3}}{3n+2}$,

(c) $x_n = \dfrac{n}{n^2+1}$,

(d) $x_n = \dfrac{5n-7}{8^n}$,

(e) $x_n = \dfrac{n^n}{(2n)!}$,

(f) $x_n = \dfrac{n+1}{n-1}$,

(g) $x_n = \dfrac{10^n}{n!}$,

(h) $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n-1} + \dfrac{1}{n}$,

(i) $x_n = \dfrac{1}{2} + \dfrac{1}{6} + \ldots + \dfrac{1}{(n-1)n} + \dfrac{1}{n(n+1)}$,

Problem 4. Consider each of the following recursive sequences. Show that they converge by using the **Monotone Convergence Theorem**. Then, find their limit $\lim_{n\to\infty} x_n$.

- $x_{n+1} = \dfrac{x_n + 1}{4}$, with $x_1 = 7$.
- $x_{n+1} = \dfrac{1}{2}x_n + 2$, with $x_1 = 1/2$.

Solution. The first terms of the sequence are

$$(x_1, x_2, x_3, \ldots) = (0.5, 2.25, 3.125, \ldots),$$

and we thus try to prove that (x_n) is increasing. Indeed, by induction, the base case is $x_1 < x_2$ true since $0.5 < 2.25$. The induction steps assumes $x_n < x_{n+1}$ and we want to show $x_{n+1} < x_{n+2}$. By the recursive definition, this latter inequality is

$$x_{n+1} < x_{n+2} \Longleftrightarrow \frac{1}{2}x_n + 2 < \frac{1}{2}x_{n+1} + 2 \Longleftrightarrow x_n < x_{n+1},$$

which is true by the induction hypothesis. Thus the sequence is increasing.

Let us prove that the sequence is bounded above, so that we can apply the Monotone Convergence Theorem and conclude (x_n) converges. Let us show that x_n is bounded above by 5 (the best upper bound is actually 4). Indeed, by induction, the base case will be $x_1 < 5$. We now assume that $x_n < 5$ and we want to prove $x_{n+1} < 5$. By the recursive definition we have

$$x_{n+1} < 5 \Longleftrightarrow \frac{1}{2}x_n + 2 < 5 \Longleftrightarrow x_n < 6,$$

which is true by the induction hypothesis. Thus the sequence (x_n) is bounded above.

By the Monotone Convergence Theorem, the sequence (x_n) is convergent. Finally, let us find its limit L. For that, substitute $L \in \mathbb{R}$ in the recursion, and we obtain:

$$L = \frac{1}{2}L + 2,$$

which leads to $L = 4$. Hence the sequence $(x_n) \to 4$, i.e. the sequence converges to 4.

- $x_{n+1} = \dfrac{1}{3 - x_n}$, with $x_1 = 2$.
- $x_{n+1} = \sqrt{3 + x_n}$, with $x_1 = 1$.
- $x_{n+1} = \sqrt{17 + x_n}$, with $x_1 = \sqrt{17}$.

- $x_{n+1} = \dfrac{x_n^2 - 63}{2}$, with $x_1 = 10$.
- $x_{n+1} = 7 - \dfrac{10}{x_n}$, with $x_1 = 4$.

Problem 5. Prove that the following numbers are **not** rational:

$$\sqrt{5}, \quad \sqrt[7]{11}, \quad \sqrt{2}+\sqrt{3}.$$

Solution (a). Let me proof that $\sqrt{5}$ is not rational. By contradiction, if $\sqrt{5} \in \mathbb{Q}$ then there would exist $p, q \in \mathbb{Z}$ such that

$$\sqrt{5} = \frac{p}{q}, \quad \text{equivalently } 5p^2 = q^2,$$

and we can assume, after clearing common factors, that $gcd(p,q) = 1$, so that they have no common divisor. Let us now look at the equality

$$5p^2 = q^2.$$

The left hand side is divisible by 5, and thus 5 divides q^2, which is the right hand side. Since 5 divides q^2, then 5 must divide q. If 5 divides q, then 5^2 divides q^2, which is the right hand side. In particular, 5^2 divides the left hand side $5p^2$. If 5^2 divides $5p^2$, we have that 5 divides p^2, and in consequence p.

The conclusion of this tongue-twisting paragraph is that 5 divides q **and** 5 divides p. This is a contradiction with our initial assumption that $gcd(p,q) = 1$, since 5 would be a common divisor.

Solution (b) Now let me show that $\sqrt[7]{11}$ is not rational by the same method. By contradiction, if $\sqrt[7]{11} \in \mathbb{Q}$ then there would exist $p, q \in \mathbb{Z}$ such that

$$\sqrt[7]{11} = \frac{p}{q}, \quad \text{equivalently } 11p^7 = q^7,$$

and we assum again that $gcd(p,q) = 1$. Let us look at the equality

$$11p^7 = q^7.$$

This time the argument sound identical, but with 11. The left hand side is divisible by 11, and thus 11 divides q^7, which is the right hand side. Since 11 divides q^7, then 11 must divide q. If 11 divides q, then 11^7 divides q^7, which is the right hand side. Now, 11^7 divides the left hand side $11p^7$. If 11^7 divides $11p^7$, we have that 11^6 divides p^7, and in consequence p. In particular, 11 divides p.

The conclusion again is that 11 divides q **and** 11 divides p. This is again contradiction with our initial assumption that $gcd(p,q) = 1$.

Solution (c) Let me finally show that $\sqrt{2}+\sqrt{3}$ is irrational. First, by the same argument as in part (a), we prove that $\sqrt{6}$ is irrational. Now, assume by contradiction that $\sqrt{2}+\sqrt{3}$ is rational. Since the product of rational numbers is rational, we have that $(\sqrt{2}+\sqrt{3})^2 = 5 + 2\sqrt{6}$, is a rational number. Now $5 + 2\sqrt{6}$ is rational if and only if $\sqrt{6}$ is rational, which is a contradiction. Hence $\sqrt{2}+\sqrt{3}$ was not rational to begin with.

Problem 6. Show that for any real number $r \in \mathbb{R}$, there exists a sequence of rational numbers $(x_n) \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = r$. In particular, notice that a sequence of rational numbers can converge to an irrational number.

Solution: For any real number $r \in \mathbb{R}$ consider, for each $n \in \mathbb{N}$, the interval $[r - \frac{1}{n}, r] \subseteq \mathbb{R}$. Apply Theorem 11.8 to this interval $[r - \frac{1}{n}, r]$ in order to obtain a rational number $q_n \in [r - \frac{1}{n}, r]$. Then $x_n = q_n$ is a sequence of rational numbers, and since $q_n \in [r - \frac{1}{n}, r]$, their limit must be r.

Problem 7. For each of the following sets X, Y and function $f : X \longrightarrow Y$, determine whether f is an **injection**, a **surjection** or a **bijection**.

(a) $f : \mathbb{N} \longrightarrow \mathbb{N}$, $f(x) = x^2$,

Solution (a). This is injective since $x^2 = y^2$ implies $|x| = |y|$, which implies $x = y$ if both numbers are natural. This is **not** surjective since 3 is not of the form $3 = x^2$ for a natural number $x \in \mathbb{N}$.

(b) $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(x) = x^2$,

Solution (b). This is **not** injective since $x^2 = y^2$ implies $|x| = |y|$, but this does **not** imply $x = y$ if are integers. Since $f(3) = f(-3) = 9$, the function is not injective. This function is also **not** surjective since again $3 \in \mathbb{Z}$ is not of the form $3 = x^2$ for an integer $x \in \mathbb{N}$.

(c) $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $f(x) = x^2$,

Solution (c). This is injective since $x^2 = y^2$ implies $|x| = |y|$, which implies $x = y$ if are positive real numbers. This function is also surjective since for any positive number $y \in \mathbb{R}^+$, the number $\sqrt{y} \in \mathbb{R}$ gives $f(\sqrt{y}) = y$.

(d) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3$,

(e) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^4 + x^2 + 3$,

(f) $f : \mathbb{Z}^{\neq 0} \longrightarrow \mathbb{Q}$, $f(x) = x^{-5}$,

(g) $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$, $f(x, y) = x \cdot y$,

(h) $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $f(x, y) = x \cdot y$,

(i) $f : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$, $f(x) = (x, x)$.

Problem 8. Determine the cardinality of each of the following sets.

(a) $X_1 = \{(x, y) : x \in \mathbb{N}, y \in \mathbb{Z}\} = \mathbb{N} \times \mathbb{Z}$,

(b) $X_2 = \{(x, y, z) : x, y, z \in \mathbb{Q}\} = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$,

(c) $X_3 = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$,

(d) $X_4 = \{x \in \mathbb{Z} : x \equiv 3 \mod 5\}$,

(e) $X_5 = \{p \in \mathbb{N} : p \text{ is a prime}\}$,

(f) $X_6 = \{x : x \in \mathbb{R} \setminus \mathbb{Q}, -5 < x < 7\} = (-5, 7) \cap \mathbb{I}$,

(g) $X_7 = \{x : x \in \mathbb{Q} \text{ or } x^2 \in \mathbb{Q} \text{ or } x^3 \in \mathbb{Q}\}$.

Answers. The sets X_1, X_2, X_4, X_5 are countable because they are subsets of countable sets by Theorem 13.14, Corollary 13.18 and the fact that the product of (finitely many) countable sets is countable. The set X_7 is countable, since it is in bijection with the union $\mathbb{Q} \cup \mathbb{Q} \cup \mathbb{Q}$, which is a finite union of countable sets, and thus it is countable.

The sets X_3 is uncountable because they admit surjections onto uncountable sets. Indeed, X_3 surjects onto $\mathbb{R}$ by mapping onto the first factor $\mathbb{R}$ in $\mathbb{R} \times \mathbb{R}$. The set X_6 is uncountable, and can be proven as follows. First, write

$$(-5, 7) = ((-5, 7) \cap \mathbb{I}) \cup ((-5, 7) \cap \mathbb{Q}),$$

and notice that $(-5, 7)$ is uncountable by Corollary 13.29. In addition, $(-5, 7) \cap \mathbb{Q}$ is countable since it is a subset of $\mathbb{Q}$. Now the union of two countable sets is countable, and in consequence $(-5, 7) \cap \mathbb{I})$ cannot be countable.

Problem 9. Define a *word* to be a finite sequence of letters, where the letters are taken from a finite set, oftentimes called the abecedary.

(a) Show that the set of all words is countable.

Solution. The set of all words is a countable union of countable sets, and thus countable.

(b) Define a *text* as a subset of the set of all words. Is the set of texts countable ?

Solution. It is uncountable because it is in bijection with the power set of a countable set, and Theorem 13.31 shows that there is no surjection of a set onto its power set.

Problem 10. Prove or disprove the following statements:

(a) Let X be a countable set, and $S \subseteq X$ a subset. Then S is itself countable,

(b) Let X, Y be countable sets, then $X \times Y$ is countable,

(c) Let X be a set and $P(X)$ its power set. If $P(X)$ is uncountable then X is countable,

(d) Let X, Y be such that $X \times Y$ is uncountable, then X **or** Y are uncountable,

(e) Let X, Y be such that $X \times Y$ is uncountable, then X **and** Y are uncountable,

(f) There exist sets X, Y and maps $f, g : X \longrightarrow Y$, such that f is an injection but *not* a surjection, **and** g is a bijection.

(g) There exist finite sets X, Y and maps $f, g : X \longrightarrow Y$, such that f is an injection but *not* a surjection, **and** g is a bijection.

Answers. $(a), (b), (d), (f)$ are true, and $(c), (e)$ and (g) are false.