University of California Davis **Name (Print):** ____________________
Differential Topology MAT-239 **Student ID (Print):** ____________________

Midterm Examination I **Oct 25 2024**
Time Limit: 50min

This examination document contains 5 pages, including this cover page, and 4 problems.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) Solve the following three parts.

   (a) (5 points) Show that
   $$\{(x, y) \in \mathbb{R}^3 : y = |x|\} \subseteq \mathbb{R}^2$$
   is not a smooth manifold in $\mathbb{R}^2$.

   (b) (10 points) Show that
   $$X = \{(x, y, z, w) \in \mathbb{R}^4 : 2x^2 + 5y^2 + 8z^2 + 3w^2 = 18\} \subseteq \mathbb{R}^4$$
   and
   $$Y = \{(x, y, z, w) \in \mathbb{R}^4 : x^3 - x - y + z^2 + w^2 = 1, z - w = 0\} \subseteq \mathbb{R}^4$$
   are smooth manifolds in $\mathbb{R}^4$.

   (c) (10 points) Consider the intersection points $p = (1, 1, 1, 1) \in X \cap Y$. Prove or disprove whether the intersection of X and Y at p is transverse.

2. (25 points) Consider the smooth manifold

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

and the restriction $f_\alpha : S^1 \longrightarrow \mathbb{R}$ of $F_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $F_\alpha(x, y) := \alpha x^2 - y$, to S^1, $\alpha \in \mathbb{R}_{\geq 0}$.

(a) (10 points) Find the critical points of $f_\alpha : S^1 \longrightarrow \mathbb{R}$, for each $\alpha \in \mathbb{R}_{\geq 0}$.

(b) (15 points) Find the values of $\alpha \in \mathbb{R}$ for which f_α is a Morse function and, for those values, compute the indices of its critical points.

3. (25 points) Consider the smooth manifold

$$X = \{A \in M_2(\mathbb{R}) : \det(A) = 1\} \subseteq \mathbb{R}^4$$

and the map $\exp : V \longrightarrow X$, $\exp(N) := e^N$, where $V := \{N \in M_2(\mathbb{R}) : \mathrm{tr}(N) = 0\} \subseteq \mathbb{R}^4$.

(a) (15 points) Show that $\exp : V \longrightarrow X$ is a local diffeomorphism at $N = 0$.

(b) (10 points) Show that $\exp : V \longrightarrow X$ is not a global diffeomorphism.

4. (25 points) For each of the sentences below, circle **the correct answer**.
(You do *not* need to justify your answer.)

   (a) (5 points) Let $f : X \longrightarrow Y$ be a smooth map and $\phi : X \longrightarrow X$ a diffeomorphism. The differential of $\phi \circ f : X \longrightarrow Y$ has the expression

   (a) $d\phi \circ df$. (b) $d(\phi^{-1}) \circ df$. (c) $df \circ d\phi$. (d) $(d\phi)^t \circ df$.

   (b) (5 points) Let $f : X \longrightarrow Y$ be a smooth map and $\phi : X \longrightarrow X$ a diffeomorphism. The Hessian $\mathrm{H}(\phi \circ f)$ of $\phi \circ f : X \longrightarrow Y$ has the expression

   (a) $\mathrm{H}(\phi)^{-1} \circ \mathrm{H}(f) \circ \mathrm{H}(\phi)$. (b) $\mathrm{H}(\phi) \circ \mathrm{H}(f) \circ \mathrm{H}(\phi)^{-1}$. (c) $\mathrm{H}(\phi)^t \circ \mathrm{H}(f) \circ \mathrm{H}(\phi)$.
   (d) None of the above.

   (c) (5 points) Let $f : X \longrightarrow Y$ be an injective immersion, then $\mathrm{im}(f)$ is a submanifold under the condition that

   (a) X is compact. (b) No condition. (c) Y is compact. (d) None of these.

   (d) (5 points) The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3 + \alpha x$ is a Morse function if

   (a) $\alpha = 0$ (b) $\alpha \neq 0$ (c) $\alpha \neq \pm 1$ (d) None of the above.

   (e) (5 points) The set of regular values of a smooth function $f : X \longrightarrow \mathbb{R}^3$ is

   (a) Discrete. (b) Measure zero. (c) Dense. (d) Empty. (e) None of the above.