**University of California Davis**
**Differential Topology MAT-239**

**Name (Print):** ____________________
**Student ID (Print):** ____________________

**Midterm Examination II**
**Time Limit: 50min**

**Nov 25 2024**

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This examination document contains 6 pages, including this cover page, and 4 problems.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| 3 | 25 | |
| 4 | 25 | |
| Total: | 100 | |

1. (25 points) Consider $\mathbb{R}^3$ with cylindrical coordinates $(r, \theta, z)$, $r \in \mathbb{R}_{\geq 0}$ and $\theta \in S^1$ so that $(r, \theta) \in \mathbb{R}^2$ are polar coordinates, and $z \in \mathbb{R}$.

   (a) (10 points) Consider $\alpha = \cos(r)dz + r\sin(r)d\theta$.
   Show that the 3-form $\alpha d\alpha \in \Omega^3(\mathbb{R}^3)$ is no-where zero.

   We have $d\alpha = -\sin(r)drdz + (\sin(r) + r\cos(r))drd\theta$ and thus

   $\alpha d\alpha = -r\sin^2(r)d\theta drdz + \cos(r)(\sin(r) + r\cos(r))dzdrd\theta = (r + \cos(r)\sin(r))(dzdrd\theta)$.

   Recall from lecture that the area form in polar coordinates $(r, \theta) \in \mathbb{R}^2$ is given by $rdrd\theta$, and thus $rdrd\theta dz$ is no-where zero.[1] Since $f'(r) = 1 + \cos(2r) \geq 0$, the function $f(r) = (r + \cos(r)\sin(r)) = r + \frac{1}{2}\sin(2r)$ is a positive function for $r > 0$ and $r + \frac{1}{2}\sin(2r) \simeq 2r$ at $r = 0$, so the form $\alpha d\alpha$ is always positive.

   (b) (5 points) Find the value $\alpha d\alpha(\partial_x, \partial_y, \partial_z)$, where $\partial_x, \partial_y, \partial_z \in T_0\mathbb{R}^3$ is the Cartesian axial basis of the tangent space of $0 \in \mathbb{R}^3$.

   By the computation i (a), the 3-form $\alpha d\alpha$ at $r = 0$ reads $2dxdydz$ in Cartesian coordinates. Therefore $\alpha d\alpha(\partial_x, \partial_y, \partial_z) = 2/3! = 1/3$.

[1] Please note that it is not zero at $r = 0$.

(c) (10 points) Let $\mathbb{R}^2_{x,y}$ have coordinates $(x,y) \in \mathbb{R}^2$. Consider $\mathbb{R}^5 = \mathbb{R}^3_{r,\theta,z} \times \mathbb{R}^2_{x,y}$ and

$$\lambda = xdy - ydx \in \Omega^1(\mathbb{R}^2).$$

Compute $\eta(d\eta)^2 \in \Omega^5(\mathbb{R}^5)$ where $\eta = \alpha + \lambda$.

We have $d\eta = d\alpha + d\lambda$ and so

$$d\eta^2 = d\alpha^2 + d\lambda^2 + d\alpha d\lambda + d\lambda d\alpha = 2d\alpha d\lambda$$

since $d\alpha^2 = d\lambda^2 = 0$, as they are 4-forms in less than 4 variables. Thus

$$\eta(d\eta)^2 = (\alpha + \lambda)2d\alpha d\lambda = 2\alpha d\alpha d\lambda = (2r + \sin(2r))(dzdrd\theta dxdy),$$

since again $\lambda d\lambda = 0$ because it is a 3-form in 2 variables.

2. (25 points) Consider $X = \mathbb{R}^3 \setminus \{0\}$ with Cartesian coordinates $(x, y, z)$, and the 2-form

$$\omega = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \cdot (xdydz + ydzdx + zdxdy) \in \Omega^2(X).$$

(a) (10 points) Show that $d\omega$=0.

Write $r^2 = x^2 + y^2 + z^2$ so $\omega = f(r)(xdydz + ydzdx + zdxdy)$ with $f(r) = r^{-3}$. Then

$$d\omega = f'(r)dr(xdydz + ydzdx + zdxdy) + f(r)(3dxdydz)$$

where $dr$ solves $2rdr = 2(xdx + ydy + zdz)$. Since $f'(r) = -3r^{-4} = -3f(r)r^{-1}$,

$$d\omega = f'(r)r^{-1}(xdx + ydy + zdz)(xdydz + ydzdx + zdxdy) + f(r)(3dxdydz) =$$

$$= f'(r)r^{-1}(r^2dxdydz) + f(r)(3dxdydz) = f'(r)rdxdydz + 3f(r)dxdydz = 0.$$

(b) (15 points) Prove that $\omega$ is not exact, i.e. $\nexists \eta \in \Omega^1(X)$ such that $d\eta = \omega$.

Consider the inclusion $i : S^2 \longrightarrow \mathbb{R}^3 \setminus \{0\}$ of the unit sphere. On the one hand, if $\omega = d\eta$ were exact, Stokes' theorem would imply that

$$\int_{S^2} i^*\omega = \int_{\partial S^2} i^*\eta = 0$$

because $S^2$ has no boundary. On the other hand, we proved in lecture that $\omega$ restricts to the volume form on $S^2$, so $i^*\omega$ is positive everywhere. Therefore

$$\int_{S^2} i^*\omega > 0.$$

This is a contradiction, so $\omega$ is not exact.

3. (25 points) Consider the 2-form $\omega = dxdy + dydz \in \Omega^2(\mathbb{R}^3)$ and the unit 2-sphere

$$S^2 = \{(\sin(\theta)\sin(\varphi), \cos(\theta)\sin(\varphi), \cos(\varphi)) \in \mathbb{R}^3 : (\theta, \varphi) \in [0, 2\pi) \times [0, \pi]\} \subseteq \mathbb{R}^3$$

parametrized with spherical coordinates $(\theta, \varphi) \in [0, 2\pi) \times [0, \pi]$. Denote by $i : S^2 \longrightarrow \mathbb{R}^3$ the inclusion map.

(a) (10 points) Compute the restriction $i^*\omega \in \Omega^2(S^2)$.

This is a direct computation:

$$i^*dx = d(\sin(\theta)\sin(\varphi)) = \cos(\theta)\sin(\varphi)d\theta + \sin(\theta)\cos(\varphi)d\varphi$$
$$i^*dy = d(\cos(\theta)\sin(\varphi)) = -\sin(\theta)\sin(\varphi)d\theta + \cos(\theta)\cos(\varphi)d\varphi$$
$$i^*dz = d(\cos(\varphi)) = -\sin(\varphi)d\varphi.$$

The resulting $i^*\omega$ in spherical coordinates follows from these identities above, e.g.

$$i^*(dydz) = (-\sin(\theta)\sin(\varphi)d\theta + \cos(\theta)\cos(\varphi)d\varphi)(-\sin(\varphi)d\varphi) = \sin(\theta)\sin^2(\varphi)d\theta d\varphi.$$

(b) (15 points) Show that $\int_{S^2} i^*\omega = 0$.

Note that $\omega$ is exact. For instance, $d\eta = \omega$ where $\eta = xdy + ydz$. By Stokes' theorem, since $S^2$ has no boundary, $\int_{S^2} i^*\omega = \int_{\partial S^2} i^*\eta = \int_{\emptyset} \eta = 0$.

4. (25 points) Let $T^2 = S^1 \times S^1 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ with coordinates $(\theta_1, \theta_2) \in T^2$. Consider the smooth map

$$f : T^2 \longrightarrow T^2, \quad f(\theta_1, \theta_2) = (4\theta_1 + 5\theta_2, 2\theta_1 + 3\theta_2),$$

and the 2-form $\eta = d\theta_1 d\theta_2 \in \Omega^2(T^2)$.

(a) (10 points) Compute the integral $\displaystyle\int_{T^2} f^*\eta$.

By the degree formula $\displaystyle\int_{T^2} f^*\eta = \deg(f) \int_{T^2} \eta$. We have

$$\int_{T^2} \eta = \int_{T^2} d\theta_1 d\theta_2 = 4\pi^2$$

and $f$ is a linear map, so that $df = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$. Thus $\deg(f) = \det(df) = 2$ and so

$$\int_{T^2} f^*\eta = \deg(f) \int_{T^2} \eta = 8\pi^2.$$

(b) (15 points) Show that $f$ is not homotopic to $g$, where

$$g : T^2 \longrightarrow T^2, \quad g(\theta_1, \theta_2) = (\theta_1 + \theta_2, \theta_1 + 23\theta_2).$$

The degree of $g$ is $\deg(g) = \det(dg) = \det \begin{pmatrix} 1 & 1 \\ 1 & 23 \end{pmatrix} = 22$. Since $\deg(f) = 2 \neq \deg(g)$ and deg is a homotopy invariant, $f$ is not homotopic to $g$. Alternatively, the degree formula gives

$$\int_{T^2} g^*\eta = \deg(g) \int_{T^2} \eta = 88\pi^2,$$

therefore $\int_{T^2} g^*\eta \neq \int_{T^2} f^*\eta$ and, as deduced by Stokes' Theorem in lecture, $f$ and $g$ cannot then be homotopic.