MAT 239: PROBLEM SET 1

DUE TO FRIDAY OCT 4 2024

ABSTRACT. This problem set corresponds to the first week of the course MAT-239 Fall 2024. It is due Friday Oct 4 at 10:00am submitted via Gradescope. Each problem worth 25 points, for a total of 100 points.

Task: Solve the *four* problems below and submit it through Gradescope by Friday Oct 4 at 10am. Be rigorous and precise in writing your solutions.

Problem 1. For each of the following maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, determine whether f is smooth and, if f is smooth, whether it is a diffeomorphism onto its image:

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$.
(2) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3 + x$.
(3) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3 - x$.
(4) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \cos(|x|)$.
(5) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x) = (x, y^2)$.
(6) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = (x^3 + xy, y)$.
(7) Discuss depending on the value of $t \in [-1, 1]$ for

$$f_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ f(x, y) = (x^3 + xy^2 - tx, y)$$

(8) Discuss depending on the value of $t \in [-1, 1]$ for

$$f_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ f(x, y) = (x^4 + xy - tx^2, y)$$

(9) $f : [0, 2\pi) \longrightarrow \mathbb{R}^2$, $f(x) = (\cos(x), \sin(x))$.

Problem 2. Find examples of the following instances:

(1) A smooth bijective map $f : \mathbb{R} \longrightarrow \mathbb{R}$ with continuous image which is not a diffeomorphism.

(2) A smooth manifold $X \subseteq \mathbb{R}^n$ and a smooth map $f : X \longrightarrow \mathbb{R}^{n+1}$ such that $f(X)$ is not a smooth submanifold.

(3) Let $C \subseteq [0, 1]$ be a closed subset (e.g. a Cantor set). A smooth function $f : [0, 1] \longrightarrow \mathbb{R}_{\geq 0}$ which vanishes exactly at C.

(4) Two diffeomorphic subsets $X, Y \subseteq \mathbb{R}^3$ such that there exists no diffeomorphism $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with $f(X) = Y$.

Problem 3. Prove or disprove whether these subsets $X \subseteq \mathbb{R}^n$ are smooth manifolds:

(1) $X = S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subseteq \mathbb{R}^{n+1}$.

(2) $X = T^n := (S^1)^n \subseteq \mathbb{R}^{2n}$.

(3) $X = \{(x, y, z) \in \mathbb{R}^3 : (9 - \sqrt{x^2 + y^2})^2 + z^2 = 1\} \subseteq \mathbb{R}^3$.

(4) $X = \{(x, y, z) \in \mathbb{R}^3 : (1 - \sqrt{x^2 + y^2})^2 + z^2 = 4\} \subseteq \mathbb{R}^3$.

(5) $X = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 - x^2 + y^2 = 0\} \subseteq \mathbb{R}^2$.

(6) $X = im(f) \subseteq \mathbb{R}^3$ where $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ is given by

$$f(t) = ((2 + \cos 3t) \cos 2t, (2 + \cos 3t) \sin 2t, \sin 3t).$$

Problem 4. Solve the following parts:

(a) Let $X = \{(a, b) \in \mathbb{R}^2 : x^3 + ax + b \text{ has a multiple root}\} \subseteq \mathbb{R}^2$. Show that X is not a manifold.

(b) Let $X = \{(a, b, c) \in \mathbb{R}^3 : x^4 + ax^2 + bx + c \text{ has a multiple root}\} \subseteq \mathbb{R}^3$. Show that X is not a manifold.

(*You might infer a general statement from* (*a*) *and* (*b*).)

(c) Show that an open subset $U \subseteq \mathbb{R}^n$ is (always) a smooth manifold.

(d) Give an example of a smooth manifold $X \subseteq \mathbb{R}^n$ and a smooth function $f : X \longrightarrow \mathbb{R}$ such that there exists two distinct values $a, b \in \mathbb{R}$ with the property that the level set $f^{-1}(a) \subseteq \mathbb{R}^n$ is a non-empty smooth submanifold but the level set $f^{-1}(b) \subseteq \mathbb{R}^n$ is not a smooth submanifold.

(e) Find two smooth manifolds $X, Y \subseteq \mathbb{R}^n$ whose intersection $X \cap Y \subseteq \mathbb{R}^n$ is not a smooth manifold.

(*Note that* (*e*) *generalizes* (*d*). *By* (*c*), *it cannot be that* X, Y *are open.*)