MAT 239: PROBLEM SET 2

DUE TO FRIDAY OCT 11 2024

Abstract. This problem set corresponds to the second week of the course MAT-239 Fall 2024. It is due Friday Oct 11 at 23:59h submitted via Gradescope. Each problem worth 25 points, for a total of 100 points.

Task: Solve the *four* problems below and submit it through Gradescope by Friday Oct 11 at 10am. Be rigorous and precise in writing your solutions.

Problem 1. (42 points) For each of the smooth manifolds $X \subseteq \mathbb{R}^N$ and points $x \in X$, compute the tangent space $T_x X$ as a linear subspace of $\mathbb{R}^N$.

(1) $X = S^2 := \{x \in \mathbb{R}^3 : |x| = 1\} \subseteq \mathbb{R}^3$ at the points: $(1,0,0)\,,(0,1,0)\,,\frac{1}{\sqrt{3}}(1,1,1)$.

(2) $X = T^2 := (S^1)^2 \subseteq \mathbb{R}^4$ at the points: $(1,0;1,0)\,,\left(0,1;\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

(3) $X = im(f) \subseteq \mathbb{R}^3$ where $f : \mathbb{R} \longrightarrow \mathbb{R}^3$ is given by

$$f(t) = (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$$

at the points $(0,0,0), (-2,-3,-9) \in X$.

(4) The genus 2 surface $X = \{(x,y,z) \in \mathbb{R}^3 : ((x^2+y^2)^2 - x^2 + y^2)^2 + z^2 = 0.02\} \subseteq \mathbb{R}^3$ at the point $(0,0,1/5\sqrt{2}) \in X$.

(5) $X = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} \subseteq \mathbb{R}^{n^2}$ at the point $A = I$, where I denotes the identity matrix and we have identified $M_n(\mathbb{R}) \cong R^{n^2}$ via the standard entries of an $(n \times n)$ matrix.

(6) $X = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} \subseteq \mathbb{R}^{n^2}$ at the points $A = I$ and $A = -I$.[1]

(7) (Optional) $X = \{A \in M_n(\mathbb{R}) : AA^t = I\} \subseteq \mathbb{R}^{n^2}$ at the point $A = I$.

Problem 2. (20 points) Solve the following parts:

(1) Prove that the map $\exp : \mathbb{R} \longrightarrow S^1$ given by $x \mapsto \exp(ix)$ is a local diffeomorphism but not a global diffeomorphism.
(Here we have identified $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.)

(2) Find all the points $(x,y) \in \mathbb{R}^2$ such that the map

$$f(x,y) = (x^3 + yx, y)$$

is *not* a local diffeomorphism.

[1] In Problem 4 below we will see that X is a smooth manifold: you may assume this for now.

(3) Consider the map
$$f_k : \mathbb{C} \longrightarrow \mathbb{C}, \quad z \mapsto z^k.$$
Show that, as a smooth map from $\mathbb{R}^2 \cong \mathbb{C}$ to $\mathbb{R}^2 \cong \mathbb{C}$, f_k is a local diffeomorphism everywhere except for the origin.

(4) Consider $X_k = \{(z,w) \in \mathbb{C}^2 : w^k + z^2 = 1\} \subseteq \mathbb{C}^2$ and the map
$$f : X_k \longrightarrow \mathbb{C}, \quad (z,w) \mapsto w.$$
Find all the points in X_k such that f is *not* a local diffeomorphism.

Problem 3. (18 points) Let $\mathfrak{so}(3) := \{V \in M_3(\mathbb{R}) : V + V^t = 0\} \subseteq \mathbb{R}^6$ be the set of skew-symmetric and $SO(3) := \{A \in M_3(\mathbb{R}) : AA^t = I\} \subseteq \mathbb{R}^6$ the set of orthogonal matrices. Consider the map
$$\exp : \mathfrak{so}(3) \longrightarrow SO(3)\,, V \mapsto \exp(V).$$

(1) Show that exp is well-defined, i.e. that its target is indeed $SO(3)$.

(2) Show that $\exp : \mathfrak{so}(3) \longrightarrow SO(3)$ is a local diffeomorphism at the origin $V = 0 \in \mathbb{R}^6$, i.e. at the zero matrix.

(3) Find a local chart centered at any matrix $A \in SO(3)$ by using (2).

Problem 4. (20 points) For this problem you can use the *regular value theorem.*[2]:

(a) Show (again!) that $S^n \subseteq \mathbb{R}^{n+1}$ is a smooth submanifold.

(b) Find all the values $a \in \mathbb{R}$ such that $X = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = a\}$ is a smooth submanifold.

(c) Show that $X = \{A \in M_n(\mathbb{R}) : \det(A) = 1\} \subseteq \mathbb{R}^{n^2}$ is a smooth submanifold.

(d) Show that $X = \{A \in M_n(\mathbb{R}) : AA^t = I\} \subseteq \mathbb{R}^{n^2}$ is a smooth submanifold.

[2]We will cover it on Monday as a consequence of the Inverse Function Theorem.