MAT 239: PROBLEM SET 4

DUE TO FRIDAY NOV 8 2024

Abstract. This problem set corresponds to early Chapter IV, the week after Midterm I, for MAT-239 Fall 2024. It is due Friday Nov 8 at 10:00am submitted via Gradescope.

Task: Solve Problems 2 and 4 below and submit them through Gradescope by Friday Nov 8 at 10:00am. Be rigorous and precise in your solutions. Please try all the problems, though they will not be graded, and work through the corresponding problems in the textbook.

Problem 1. Given the following differential forms, compute the following:

(1) Let $\omega = dx_1dx_2 + dx_3dx_4 \in \Omega^2(\mathbb{R}^4)$, compute its powers $\omega^n \in \Omega^{2n}(\mathbb{R}^4)$ for all $n \in \mathbb{N}$.
(2) As above, let $\omega = dx_1dx_2 + dx_3dx_4 \in \Omega^2(\mathbb{R}^4)$. Consider a linear map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, given in block matrix form

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(\mathbb{R}), \quad A, B, C, D \in M_2(\mathbb{R}).$$

Find necessary and sufficient conditions on $A, B, C, D \in M_2(\mathbb{R})$ such that $f^*\omega = \omega$.

(3) Let $\alpha = dz - ydx \in \Omega^1(\mathbb{R}^3)$, compute $d\alpha$ and the wedge product $\alpha d\alpha \in \Omega^3(\mathbb{R}^3)$.
(4) Let $\alpha = dz - y_1dx_1 - y_2dx_2 \in \Omega^1(\mathbb{R}^5)$, compute the forms $d\alpha \in \Omega^2(\mathbb{R}^3)$, $d\alpha^2 \in \Omega^4(\mathbb{R}^3)$ and the wedge product $\alpha d\alpha^2 \in \Omega^5(\mathbb{R}^3)$.
(5) Consider the following three 1-forms

$$\omega_i = x_1dx_2 - x_2dx_1 \in \Omega^1(\mathbb{R}^3),$$
$$\omega_j = x_2dx_3 - x_3dx_2 \in \Omega^1(\mathbb{R}^3),$$
$$\omega_i = x_3dx_1 - x_1dx_3 \in \Omega^1(\mathbb{R}^3).$$

Compute the wedge products $\omega_i\omega_j$, $\omega_j\omega_k$, $\omega_k\omega_i$ and $\omega_i\omega_j\omega_k$.

Problem 2. Consider the 1-form

$$\alpha = \frac{1}{2}\sum_{i=1}^{n}(x_idy_i - y_idx_i) \in \Omega^1(\mathbb{R}^{2n})$$

and the vector field

$$V = \frac{1}{2}\sum_{i=1}^{n}(x_i\partial_{x_i} + y_i\partial_{y_i}) \in \mathfrak{X}(\mathbb{R}^{2n})$$

(1) Compute $d\alpha(V) \in \Omega^1(\mathbb{R}^{2n})$ and its restriction $\eta = d\alpha(V)|_{S^{2n-1}}$ to $S^{2n-1} \subseteq \mathbb{R}^{2n}$.
(2) Compute the $(2n-1)$-form $\eta d\eta^{n-1} \in \Omega^{2n-1}(S^{2n-1})$.
(3) Given the vector field

$$R = \frac{1}{2}\sum_{i=1}^{n}(x_i\partial_{y_i} - y_i\partial_{x_i}) \in \mathfrak{X}(\mathbb{R}^{2n}).$$

Show that R is tangent to S^{2n-1}, and thus R restricts to a vector field $R \in \mathfrak{X}(S^{2n-1})$.
(4) Entirely restricted to S^{2n-1}, show that $\alpha(R) = 1$ and $d\alpha(R) = 0$.
(5) Let $\varphi_t : S^{2n-1} \longrightarrow S^{2n-1}$ be the time-t flow of the vector field R. Compute the pull-back $\varphi_t^*\alpha$.

Problem 3. Let $U \subseteq \mathbb{R}^n$ be an open subset. Solve the following parts:

(a) Suppose $\eta \in \Omega^k(U)$ exact and $\omega \in \Omega^l(U)$ closed. Show that $\eta\omega \in \Omega^{k+l}(U)$ is exact.
(b) Suppose $\omega \in \Omega^k(U)$ is closed, $k \geq 1$. Show that for any point $p \in U$ there exists an open neighborhood $V \subseteq U$ of p such that $\omega|_V$ is exact.

Problem 4. Let $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, $i \in [1, n]$, and consider the rectangle

$$R = (a_1, b_1) \times \dots (a_n, b_n) \subseteq \mathbb{R}^n.$$

(1) Given a differential form $\omega \in \Omega^n(R)$, find an explicit smooth function

$$g : R \longrightarrow \mathbb{R},$$

in terms of ω, such that $\omega = d(g(x_1, \dots, x_n)dx_2 \dots dx_{n-1})$.
(In particular ω must be exact.)

(2) Given a closed differential form $\omega \in \Omega^k(R)$, $k \in [1, n]$, show that ω is exact.

Problem 5. Let $(u_i, \varphi_i, z) \in \mathbb{R}^{2n-2} \times \mathbb{R}$ be cylindrical coordinates, so $(u_i, \varphi_i) \in \mathbb{R}^2$ are polar coordinates for $i \in [1, n-1]$. Consider a diffeomorphism $h : \mathbb{R} \longrightarrow \mathbb{R}$ and the 1-form

$$\alpha = dz - \sum_{i=1}^{n-1} u_i d\varphi_i \in \Omega^1(\mathbb{R}^{2n-1}).$$

Show that

$$\Phi_h : \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}^{2n-1}, \quad \Phi_h(u_i, \varphi_i, z) = (h'(z)u_i, \varphi_i, h(z))$$

is a diffeomorphism and that $\Phi_h^*\alpha$ is proportional to α by a non-zero function $f \in C^\infty(\mathbb{R}^{2n-1})$.

Problem 6. Consider the special orthogonal group

$$SO(n) := \{A \in M_{n\times n}(\mathbb{R}) : A \cdot A^t = \mathrm{Id}, \quad \det(A) = 1\}.$$

Let $g \in SO(n)$ and consider the diffeomorphism

$$L_g : SO(n) \longrightarrow SO(n), \quad L_g(A) = g \cdot A.$$

Show that the 1-form $\eta = A^{-1}dA \in \Omega^1(SO(n))$ satisfies $L_g^*\eta = \eta$.