MAT 239: PROBLEM SET 5

DUE TO FRIDAY NOV 15 2024

Abstract. This problem set corresponds to later Chapter IV (4.4 and 4.5), for MAT-239 Fall 2024. It is due Friday Nov 15 at 23:59pm submitted via Gradescope.

Task: Solve Problems 4 and 5 below and submit them through Gradescope by Friday Nov 15 at 23:59pm. Be rigorous and precise in your solutions. Please try all the problems, though they will not be graded, and work through the corresponding problems in the textbook.

Problem 1. Solve the following parts:

(1) Let $\eta = ydx \in \Omega^1(\mathbb{R}^2)$ and $i : S^1 \longrightarrow \mathbb{R}^2$ the inclusion of the unit circle. Find $\int_{S^1} i^*\eta$.
(2) Show that a 1-form $\eta \in \Omega^1(S^1)$ is exact if and only if $\int_{S^1} \eta = 0$.

Problem 2. Let X be an n-dimensional smooth manifold and $Z \subseteq X$ a closed subset. Suppose $Z \subseteq X$ is measure zero.[1] Show that

$$\int_X \omega = \int_{X\setminus Z} \omega, \quad \forall \omega \in \Omega^n(X).$$

Problem 3. Let $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4z\} \subseteq \mathbb{R}^3$, $i : X \longrightarrow \mathbb{R}^3$ the defining inclusion, and $\omega = zdxdy$. Compute $\int_X \omega$.

Problem 4. Consider the smooth map

$$i : D \longrightarrow \mathbb{R}^4, \quad i(x, y) = (x, x - y, 3 - x + xy, -3y),$$

where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$.

(1) Compute the integral $\int_D i^*\omega$ where

$$\omega = -x_2dx_1dx_3 + x_4dx_3dx_4.$$

(2) Consider $\eta = x_3^2dx_1$, find the value of $\int_D i^*d\eta$.
(3) Directly compute $\int_{\partial D} i^*\eta$, showing it coincides with (2).

Problem 5. Suppose $f, g : S^1 \longrightarrow \mathbb{R}^3$ are smooth embeddings with disjoint images. Consider the smooth map

$$F_{f,g} : S^1 \times S^1 \longrightarrow S^2, \quad (\theta_1, \theta_2) \mapsto \frac{f(\theta_1) - g(\theta_2)}{|f(\theta_1) - g(\theta_2)|}$$

and the pull-back $\omega \in \Omega^2(S^2)$ of $zdxdy+xdydz+ydzdx \in \Omega^2(\mathbb{R}^3)$ to the unit sphere $S^2 \subseteq \mathbb{R}^3$.

(1) Show that $\int_{S^2} \omega = 4\pi$.
(2) Show that the integral

$$L(f, g) = \frac{1}{4\pi} \int_{S^2} F_{f,g}^*\omega$$

is an integer number.
(3) Suggest a geometric meaning for that integer number.

[1]By definition, $Z \subseteq X$ is measure zero if it is a measure zero set for any (Euclidean) local chart in X.

Problem 6. Consider the 4-dimensional manifold

$X = S^2 \times S^2 = \{(x_1, x_2, x_3; y_1, y_2, y_3) \in \mathbb{R}^6 : x_1^2 + x_2^2 + x_3^2 = 1, y_1^2 + y_2^2 + y_3^2 = 1\} \subseteq \mathbb{R}^3 \times \mathbb{R}^3$.

Let $\omega \in \Omega^2(S^2)$ be a volume form and define $\eta = \omega \oplus (-\omega) \in S^2 \times S^2$. Compute $\int_K \eta$ where

$$K = \{(x, y) \in S^2 \times S^2 : x_3 + y_3 = 0, x \cdot y = -0.5\}.$$

Problem 7. Consider the special orthogonal group

$$SO(3) := \{A \in M_{3\times 3}(\mathbb{R}) : A \cdot A^t = \mathrm{Id}, \quad \det(A) = 1\}$$

and identify the tangent space at the identity via

$$T_{Id}SO(3) = \{V \in M_{3\times 3}(\mathbb{R}) : V + V^t = 0, \quad \mathrm{tr}(V) = 0\}.$$

Let $\omega \in \Omega^3(SO(3))$ be the unique 3-form which is left-invariant, i.e. invariant under the all diffeomorphisms

$$L_g : SO(3) \longrightarrow SO(3), \quad L_g(A) = g \cdot A, \quad g \in SO(3),$$

and it is given by the multilinear 3-form

$$\omega(v_1, v_2, v_3) = \mathrm{tr}([v_1, v_2]v_3), \quad \forall v_1, v_2, v_3 \in T_{Id}SO(3),$$

where $[v_1, v_2] = v_1v_2 - v_2v_1$ is the matrix commutator. Compute the integral $\int_{SO(3)} \omega$.

Problem 8. Let $(z, w) \in \mathbb{C}^2$ be complex coordinates and consider the real 2-form $\omega = \Re(dzdw)$ given by the real part. Consider the real smooth surfaces

$$X_k = \{(z, w) \in \mathbb{C}^2 : z^k + w^2 = 1\} \subseteq \mathbb{C}^2.$$

Compute the integrals $\int_{X_k} \omega$.

Problem 9. Let $T^n = S^1 \times \dots S^1 \subseteq \mathbb{R}^{2n}$ be the n-dimensional torus. Compute the de Rham cohomology groups $H^*_{dR}(T^n, \mathbb{R})$.