MAT 239: PROBLEM SET 6

Abstract. These are practice problems for Chapter III, for MAT-239 Fall 2024.

Recall that $\chi(X) := I(\Delta, \Delta)$ denotes the Euler characteristic of a compact oriented manifold X, where I is the oriented intersection index and $\Delta \subseteq X \times X$ the diagonal submanifold.

Problem 1. Consider the 2-torus $T^2 = S^1 \times S^1$ with coordinates $(\theta_1, \theta_2) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. Let $C_n = \{\theta_2 = n\theta_1\} \subseteq T^2$ be oriented as the graph of

$$f_n : S^1 \longrightarrow S^1, \quad f_n(\theta) = n\theta, \quad n \in \mathbb{N}.$$

Compute the oriented intersection index $I(C_n, C_m)$ for any $n, m \in \mathbb{N}$.

Problem 2. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be two compact oriented manifolds and

$$X \times Y = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in X, y \in Y\} \subseteq \mathbb{R}^n \times \mathbb{R}^m$$

its Cartesian product. Show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Problem 3. Consider the special orthogonal group

$$SO(n) := \{A \in M_{n\times n}(\mathbb{R}) : A \cdot A^t = \mathrm{Id}, \quad \det(A) = 1\}.$$

Show that $\chi(SO(n)) = 0$ for all $n \in \mathbb{N}$.

Problem 4. Let $f : S^n \longrightarrow S^n$ be a smooth map with degree $\deg(f) \in \mathbb{Z}$ and Lefschetz number $L(f)$. Show that

$$L(f) = 1 + (-1)^{n+1} \deg(f).$$

In particular, any $f : S^n \longrightarrow S^n$ with $\deg(f) \neq (-1)^n$ must have a fixed point.

Problem 5. Consider the smooth map

$$f : S^2 \longrightarrow S^2, \quad z \mapsto 2\frac{z^2}{|z|}$$

where $S^2 = \mathbb{C} \cup \{\infty\}$ is considered as the Riemann sphere. Show that the Lefschetz number of its nth power $f^n : S^2 \longrightarrow S^2$ is $L(f^n) = 2^n + 1$.

Problem 6. Prove the fundamental theorem of algebra as a consequence of the Lefschetz fixed point theorem.

Hint: Let $A \in GL_n(\mathbb{C})$ act on a $\mathbb{C}$-vector space and compute the Lefschetz number of such a diffeomorphism on its projectivization.