MAT 21C: SOLUTIONS TO PROBLEM SET 2

Problem 1. From "Exercises 10.2" in textbook, solve **39, 41, 44, 45, 50, 53, 58, 81, 85, 86** (each item is worth 2 points) and From "Exercises 10.3" in textbook, solve **5, 6, 8, 21, 29** (each item is worth 1 points).

Solutions to the odd numbered exercises can be found in the textbook.

Problem 2. From the textbook, solve:

- In "Exercises 10.4", **11, 12, 19, 26, 27** (each item is worth 2 points)
- In "Exercises 10.5", **1,4,6,9,11,14** (each of these items is worth 2 points)
- In "Exercises 10.6", **2, 27, 35** (worth 1 point).

Solutions to the odd numbered exercises can be found in the textbook.

Problem 3. For each statement, justify whether they are true or explain why they are false (providing a counter-example). Each item is worth 5 points.

(a) If $(a_n) \to 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges.

This is false. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, e.g. apply the integral test, or it is a p-series with $p = 1$. That said, the series $a_n = \dfrac{1}{n}$ converges to 0.

(b) If the root test is inconclusive, then the ratio test is inconclusive.

This is true. The root test is strong than the ratio test. In other words if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1$.

(c) Let (a_n) be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous positive decreasins function of x for all $x \geq 1$. If the series $\sum_{n=1}^{\infty} a_n$ converges then we have the equality

$$\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)dx.$$

This is not true. The integral test only says that the two sides of the equality behave the same, *not* that they are the same.

An explicit example is given by the geometric series with $r = 1/2$, i.e. $a_n = 2^{-n}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \neq \int_1^{\infty} \frac{1}{2^x} dx,$$

since the right hand side has value $[\ln(2)2^{-x}]_1^{\infty} = \frac{1}{\ln(4)}$.

(d) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} |a_n|$ converges.

This is false. Choose $a_n = \frac{(-1)^n}{n}$ so $|a_n| = \frac{1}{n}$. Then the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by the alternating series test. However the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, again by the integral test or p-series test with $p = 1$.

(e) There are series $\sum_{n=1}^{\infty} a_n$ for which the integral test determines convergence but the root test does not.

This is true. Choose the p-series with $p = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Then the root test gives a limit of 1, as $\sqrt[n]{n^2} \to 1$, and so it does not decide. The integral test for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ yields the integral

$$\int_1^{\infty} \frac{1}{x^2} < \infty,$$

so the integral test decides convergence.

Problem 4. In this problem we study the convergence of the series $S := \sum_{n=1}^{\infty} e^{-n^2}$ from the perspective of the different tests. Each item is worth 5 points.

(i) Show that $\sum_{n=1}^{\infty} e^{-n}$ converges and its limit is

$$\sum_{n=1}^{\infty} e^{-n} = \frac{1}{e-1}.$$

The root test gives

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|e^{-n}|} = \lim_{n\to\infty} e^{-1} < 1$$

and thus implies convergence. Alternatinvely, this is a geometric series with $r = e^{-1}$ and since $e^{-1} < 1$ it converges.

The limit of a geometric series with $r = e^{-1}$ is

$$\sum_{n=0}^{\infty} e^{-n} = r^n = \frac{1}{1-r} = \frac{1}{1-e^{-1}} = \frac{e}{e-1}.$$

So

$$\sum_{n=1}^{\infty} e^{-n} = \frac{e}{e-1} - 1 = \frac{1}{e-1}.$$

(ii) Deduce from (i) that S is convergent by comparing it to $\sum_{n=1}^{\infty} e^{-n}$.

The comparison test asks to compare via the limit

$$\lim_{n\to\infty} \frac{e^{-n^2}}{e^{-n}} = 0.$$

Thus S converges if the series in Part (i) converges.

(iii) Use the integral test to show that S converges.

Hint: You may use the beautiful equality $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

The integral test says that S converges if and only if the integral

$$\int_1^{\infty} e^{-x^2} dx$$

is finite. Since

$$0 < \int_1^{\infty} e^{-x^2} dx < \int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \cdot \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

the integral converges. Hence S converges.

(iv) Use the ratio test to show that S converges.

The terms are $a_n = e^{-n^2}$. The ratio test asks us to compute the limit

$$\lim_{n\to\infty} |a_{n+1}|/|a_n| = \lim_{n\to\infty} e^{-(n+1)^2}/e^{-n^2} = \lim_{n\to\infty} e^{-(n+1)^2+n^2} =$$

$$= \lim_{n\to\infty} e^{-n^2-2n-1+n^2} = \lim_{n\to\infty} e^{-2n-1} = 0.$$

Since this limit exists and it is less than 1, the ratio test shows that S converges.

(v) Use the root test to show that S converges.

The terms are $a_n = e^{-n^2}$. The root test requires us to compute the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{e^{-n^2}} = \lim_{n\to\infty} \sqrt[n]{e^{-n^2}} = \lim_{n\to\infty} e^{-n} = 0.$$

Since this limit exists and it is less than 1, the root test shows that S converges.

For the record, the exact value of $\sum_{n=1}^{\infty} e^{-n^2}$ is actually $(1+\vartheta_3(0,e^{-1}))/2$, where ϑ_3 is a Jacobi theta function, which encodes things such as heat dispersion, the translational partition function for an ideal gas or how natural numbers can be expressed as sums of (four) squares.