University of California Davis **Name (Print):** ____________________
Algebraic Topology MAT 215B **Student ID (Print):** ____________________

Final Examination **June 7 2024@9am**
Time Limit: Due 6/11@9pm

This examination document contains 9 pages, including this cover page, and 8 problems.

Task: **Solve three** of the problems below. You may choose which three problems.

You must show your work on each chosen problem. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
7	100	
8	100	
Total:	800	

1. (100 points) Let $K \subseteq S^3$ be the image of a smooth embedding $S^1 \to S^3$.

   (a) (70 points) Compute the homology of the complement $H_*(S^3 \setminus K, \mathbb{Z})$.

   (b) (30 points) Give an example of two such embedded 1-spheres $K_1, K_2 \subseteq S^3$ such that $S^3 \setminus K_1$ is *not* homotopic to $S^3 \setminus K_2$.

2. (100 points) Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, act on the 5-sphere

$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$$

via the formula:

$$\mathbb{Z}_n \times S^5 \to S^5, \quad (1; z_1, z_2, z_3) \mapsto (e^{2\pi i/n} \cdot z_1, e^{2\pi i/n} \cdot z_2, e^{2\pi i/n} \cdot z_3),$$

where $1 \in \mathbb{Z}_n$ is a generator. Let $X_n := S^5/\sim$ be the orbit space of this $\mathbb{Z}_n$-action, endowed with the quotient topology.

(a) (40 points) Show that

$$H_*(X_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 5 \\ \mathbb{Z}_n & \text{if } * = 1, 3 \\ 0 & \text{if } * = 2, 4. \end{cases}$$

(b) (30 points) For any $n, m \in \mathbb{N}$, compute the cohomology groups $H^*(X_n; \mathbb{Z}_m)$ with coefficients in the Abelian group $\mathbb{Z}_m$.

(c) (30 points) For any $n, m \in \mathbb{N}$, compute the homology groups $H_*(X_n \times X_m; \mathbb{Z})$.

3. (100 points) Let $T^3 = (S^1)^3$ be the 3-torus and consider the continuous map

$$f : T^3 \to T^3, \quad f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 5 & 4 & -2 \\ 4 & 4 & -3 \\ -4 & -3 & 4 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

Consider the smooth 3-dimensional manifold

$$M(f) := (T^3 \times [0,1])/\sim \qquad \text{where} \quad (f(x), 0) \sim (x, 1) \quad \text{if } x \in T^3.$$

(a) (50 points) Compute the homology groups $H_*(M(f); \mathbb{Z})$.

(b) (20 points) Is the map $f : T^3 \to T^3$ homotopic to the identity?

(c) (20 points) Show that $f : T^3 \to T^3$ has at least one fixed point.

(d) (10 points) Suppose that $g : T^3 \to T^3$ is a continuous map such that $H_*(g) = H_*(f)$, as endomorphisms of $H_*(T^3)$. Does g necessarily have a fixed point?

4. (100 points) Let $\mathcal{C}$ be the Abelian category of Abelian groups. Solve the following parts.

   (a) (40 points) Give an example of two functors $F, J : \mathcal{C} \to \mathcal{C}$ such that F is left-exact but not right-exact and J is right-exact but not left-exact.

   (b) (30 points) Give an example of a functor $H : \mathcal{C} \to \mathcal{C}$ such that H is not left-exact nor right-exact but for every short exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

   the complex

$$H(G_1) \to H(G_2) \to H(G_3)$$

   is exact.

   (c) (30 points) Give an example of a functor $H : \mathcal{C} \to \mathcal{C}$ such that H is not left-exact nor right-exact and there exists a short exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

   such that the complex

$$H(G_1) \to H(G_2) \to H(G_3)$$

   is *not* exact.

5. (100 points) Let k be a field, $R = k[x_1, \ldots, x_n]$ the polynomial algebra in n generators $x_1, \ldots, x_n$ of degree 1. Consider k as an R-module under the map $R \to k$ that quotients by the ideal $(x_1, \ldots, x_n)$.

   (a) (50 points) Show that the Ext_R groups of k with k are, as k-vector spaces,

   $$\mathrm{Ext}^i_R(k, k) \cong k^{\binom{n}{i}}, \quad i \in \mathbb{N},$$

   where $\binom{n}{i}$ denotes n choose i. (In particular, for $n < i$, $\mathrm{Ext}^i_R(k, k) = \{0\}$.)

   (b) (50 points) Let $Q = E[y_1, \ldots, y_n]$ be the exterior algebra in $y_1, \ldots, y_n$, each y_i in degree 1. Consider k as an Q-module under the map $Q \to k$ that quotients by the ideal $(y_1, \ldots, y_n)$. Compute the groups $\mathrm{Ext}^i_Q(k, k)$ for all $i \in \mathbb{N}$.

6. (100 points) Let R be a commutative ring.

   (a) (50 points) Suppose that A, B are R-modules. Show that

   $$\mathrm{Tor}_i^R(A,B) \cong \mathrm{Tor}_i^R(B,A)$$

   for all $i \in \mathbb{N}$.

   (b) (50 points) Let $I, J \subseteq R$ be ideals. Considering $R/I, R/J \in R$-mod, show that

   $$\mathrm{Tor}_0^R(R/I, R/J) \cong R/(I+J), \quad \mathrm{Tor}_1^R(R/I, R/J) \cong (I \cap J)/(I \cdot J).$$

7. (100 points) Consider the complex projective plane $\mathbb{CP}^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. Consider the smooth submanifold

$$C = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : z_0^2 + z_1^2 + z_2^2 = 0\} \subseteq \mathbb{CP}^2.$$

(a) (30 points) Show that C is diffeomorphic to S^2.

(b) (40 points) Let $U \subseteq \mathbb{CP}^2$ be an arbitrarily small (but fixed) open neighborhood of C, so that the inclusion $C \subseteq U$ is a homotopy equivalence. Compute the homology groups of the boundary of U, i.e. compute $H_*(\partial U; \mathbb{Z})$.

(c) (30 points) Compute the homology groups of the complement, i.e. $H_*(\mathbb{CP}^2 \setminus C; \mathbb{Z})$.

(d) (0 points) (*Optional*) Consider the surface

$$C_d = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : z_0^d + z_1^d + z_2^d = 0\} \subseteq \mathbb{CP}^2.$$

Compute $H_1(\mathbb{CP}^2 \setminus C_d; \mathbb{Z})$.

8. (100 points) Consider the following 3-dimensional smooth submanifold:

$$\Sigma := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^3 + z_3^5 = 0\} \cap S^5,$$

where $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ is the unit 5-sphere in $\mathbb{C}^3 = \mathbb{R}^6$.

(a) (80 points) Show that $H_*(\Sigma; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$, where S^3 is the 3-dimensional sphere.

(b) (20 points) Show that Σ is not diffeomorphic to S^3.