MAT 215B: PROBLEM SET 3

NO DUE DATE

Task: Solve some of the following problems. Be rigorous and precise in writing your solutions. These problems are for practice and do not need to be submitted.

Problem 1. (**Properties of Ext**). Let G, H be finitely generated Abelian groups.

(a) Show that $Hom(-, G)$ is right exact, i.e. if $A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, then $Hom(A, G) \longleftarrow Hom(B, G) \longleftarrow Hom(C, G) \longleftarrow 0$ is exact.

(b) Give an example which shows that $Hom(-, G)$ is *not* left exact.

(c) Show that $Ext^0(H, G) = Hom(H, G)$.

(d) Show that $Ext^i(H, G) = 0$ for $i \geq 2$.

(e) Give two examples of distinct Abelian groups H_1, H_2 such that

$$Ext^0(H_1, \mathbb{Z}) = Ext^0(H_2, \mathbb{Z}).$$

Thus, $Ext^0(H, \mathbb{Z})$ does *not* in general recover the Abelian group H.

Problem 2. (**Homological Algebra of Ext**). Let $0 \longrightarrow A \longrightarrow B \longrightarrow C$ be a short exact sequence of Abelian groups, and G an Abelian group. Show that there exists a long exact sequence

$$0 \longrightarrow Hom(C, G) \longrightarrow Hom(B, G) \longrightarrow Hom(A, G) \stackrel{\delta}{\longrightarrow} Ext^1(C, G) \longrightarrow$$
$$\longrightarrow Ext^1(B, G) \longrightarrow Ext^1(A, G) \longrightarrow 0.$$

That is, similar to the homology functor H_*, applying Ext^* to a short exact sequence yields a long exact sequence of Ext groups. (Recall that $Ext^0(H, G) = Hom(H, G)$, as you proved in Problem 1.(c), and, $Ext^i(H, G) = 0$ for $i \geq 2$, after Problem 1.(d). Thus this long exact sequence is a sequence of *all* Ext groups.)

Problem 3. (**Computations of Ext I**). All groups H_1, H_2, G_1, G_2, H, G in this problem are finitely generated Abelian groups.

(a) Show that $Ext^*(H_1 \oplus H_2, G) = Ext^*(H_1, G) \times Ext^*(H_2, G)$.

(b) Prove that $H^*(X \vee Y, G) \cong H^*(X, G) \times H^*(Y, G)$.

(b) Is $Ext^*(H, G_1 \times G_2) = Ext^*(H, G_1) \times Ext^*(H, G_2)$ true ?

(c) Show that $Ext^1(H, G) = 0$ if H is a free group.

(d) Find the cohomology groups $H^*(S^n, G)$ with coefficients in a finitely generated group Abelian G.

(e) Find the cohomology groups $H^*(\mathbb{CP}^n, G)$.

Problem 4. (**Computations of Ext II**) Solve the following parts:

(a) Compute $Ext^*(H, \mathbb{Z})$ for all finitely generated Abelian groups H. In particular, conclude that the graded groups $Ext^*(H, \mathbb{Z})$ uniquely determine H.

(b) Show that $Ext^0(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_d$ and $Ext^1(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_d$, where $d = gcd(n, m)$.

(c) Find the cohomology groups $H^*(\mathbb{RP}^n, \mathbb{Z})$ with coefficients in $\mathbb{Z}$.

(d) Compute the cohomology groups $H^*(\mathbb{RP}^n, \mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$.

(e) Which groups provide more information $H^*(\mathbb{RP}^n, \mathbb{Z}_2)$ or $H^*(\mathbb{RP}^n, \mathbb{Z}_3)$?

(f) For each odd $n \in \mathbb{N}$, give an example of a topological space X such that the cohomology groups $H^*(X, \mathbb{Z}_3) \cong H^*(S^n, \mathbb{Z}_3)$ are isomorphic, but X is not homotopy equivalent to S^n.

Problem 5. (**Group Extensions**) Let A, B be two $\mathbb{Z}$-modules, an *extension* of B by A is the data of a group G **and** a short exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0.$$

The set of all extensions of B by A is denoted $\mathcal{E}(B, A)$.

(a) Show that $G \cong A \oplus B$ if B is free, i.e. the only extensions of the Abelian groups $\mathbb{Z}^r$ are by direct sums.

(b) Show that $\mathcal{E}(\mathbb{Z}_p, \mathbb{Z}_p)$ contains p elements, if p is a prime.

(c) Given an extension $0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0$, consider the induced map $Hom(A, A) \xrightarrow{\delta} Ext^1(B, A)$ constructed above. Show that

$$\chi : \mathcal{E}(B, A) \longrightarrow Ext^1(B, A),$$
$$\chi\,(0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0) = \delta(id_A),$$

is a bijection.

(d) Conclude that there are exactly $\mathbb{Z}_{gcd(n,m)}$ extensions of $\mathbb{Z}_n$ by $\mathbb{Z}_m$.

Problem 6. (**A geometric Meaning of Ext**) This problem is only optional, in case you are taking an introductory class in Algebraic Geometry.

(a) Let $X = \mathrm{Spec}(A)$ be an affine variety, $x \in X$ a closed point given by a maximal ideal $\mathfrak{m} \subseteq A$, and $k(x)$ its residue field. Show that $Ext^1_X(k(x), k(x))$ is isomorphic to the tangent space T_xX of X at x.

Since $Ext^1_X(k(x), k(x))$ is isomorphic to $Ext^1_A(A/\mathfrak{m}, A/\mathfrak{m})$, and T_xX is, by definition, $Hom_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, A/\mathfrak{m})$, the above statement is equivalent to

$$Ext^1_A(A/\mathfrak{m}, A/\mathfrak{m}) \cong Hom_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, A/\mathfrak{m}).$$

(b) Let $E \longrightarrow X$ be a coherent sheaf (e.g. a vector bundle) over a smooth algebraic variety X, and $\mathcal{M}_X$ the moduli space of coherent sheaves. Give an heuristic explanation for the isomorphism $Ext^1(E, E) \cong T_E\mathcal{M}_X$.

Problem 7. (**Künneth Formula I**). Solve the following parts.

(a) Compute the cohomology $H^*(S^n \times S^m, \mathbb{Z})$ for all $n, m \in \mathbb{N}$.

(b) Show that the cohomology $H^*(T^n, \mathbb{Z})$ is isomorphic to the exterior algebra on n elements. In particular, show that

$$\operatorname{rank}_{\mathbb{Z}}(H^k(T^n, \mathbb{Z})) = \binom{n}{k}.$$

(c) Let X be a CW complex. Compute the cohomology of $H^*(X \times S^1, \mathbb{Z})$ in terms of $H^*(X)$.

Problem 8. (**Künneth Formula II**). A real division algebra structure on $\mathbb{R}^n$ is a bilinear multiplication map

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that $ax = b$ and $xa = b$ are uniquely solvable whenever $a \neq 0$.

(a) Show that $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^4$ admit an real division algebra structure.

(b) Suppose $(\mathbb{R}^n, \cdot)$ is a real division algebra. Show that the map

$$g : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}, \quad g(x, y) = \frac{x \cdot y}{|x \cdot y|}$$

defined by restricting the division algebra product to the unit sphere (and normalize), descends to a map $\widetilde{g} : \mathbb{RP}^n \times \mathbb{RP}^n \longrightarrow \mathbb{RP}^n$.

(c) (Assuming knowledge of cup product or ring structure on cohomology.) Let $\alpha \in H^1(\mathbb{RP}^{n-1}, \mathbb{Z}_2)$ be the generator. Show that the pull-back $\widetilde{g} * (\alpha^n)$ of the map $\widetilde{g}$ in Part (b) vanishes, i.e. $\widetilde{g} * (\alpha^n) = 0$.

(d) Show that if $(\mathbb{R}^n, \cdot)$ is a real division algebra, then $n = 2^k$ for some $k \in \mathbb{N}$.

Problem 9. Let $\mathcal{F} \in \mathrm{Sh}(S^1)$ be a locally constructible sheaf on S^1 and k a field. Suppose that the stalk $\mathcal{F}_x$ is an n-dimensional k-vector space and $M \in \mathrm{Aut}(\mathcal{F}_x)$ is its monodromy. Compute the sheaf cohomology $H^*(S^1; \mathcal{F})$.

Problem 10. Let X be a smooth manifold.

(1) Show that the sheaf of smooth differential forms Ω^*_X on the smooth manifold X is an acyclic resolution of the constant sheaf $\underline{\mathbb{R}}$ on X.

(2) Prove that $H^*_{dR}(X, \mathbb{R}) \cong H^*(X; \underline{\mathbb{R}})$, i.e. deRham cohomology is isomorphic to the sheaf cohomology of the constant sheaf $\underline{\mathbb{R}}$.