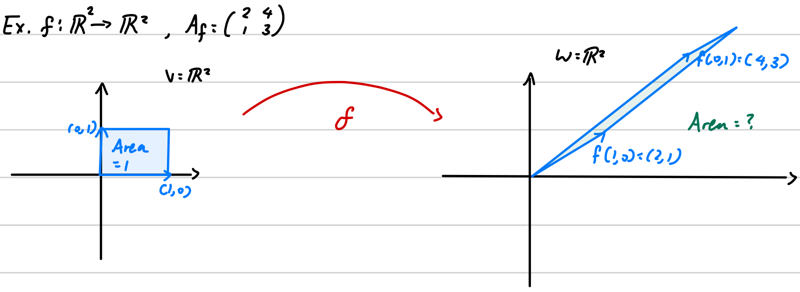
The determinant

Ex. $f: \mathbb{R}^2 \to \mathbb{R}^2$, $A_f = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$



Question: What is area of green?

Def: determinant

> Let $f: V \to W$ be a linear map
>
> Then the determinant of f, or $\det(f)$, is the area of the parallelogram spanned by $\underline{f(1,0,0,\dots,0),\ f(0,1,\dots,0),\ \dots,\ f(0,0,\dots,1)}$
>
> area ↓ signed area   ↓ standard axis basis

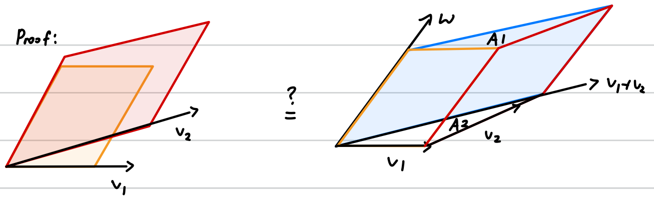
Properties:

① $\mathrm{Area}(v_1,w) + \mathrm{Area}(v_2,w) = \mathrm{Area}(v_1+v_2,w)$, i.e. $\boxed{\text{bilinearity}}$

$$\det(v_1,w) = \begin{vmatrix} v_{11} & w_1 \\ v_{12} & w_2 \end{vmatrix}, \quad \det(v_2,w) = \begin{vmatrix} v_{21} & w_1 \\ v_{22} & w_2 \end{vmatrix}, \quad \det(v_1+v_2,w) = \begin{vmatrix} v_{11}+v_{21} & w_1 \\ v_{12}+v_{22} & w_2 \end{vmatrix}$$

(e.g. $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = 0$, $\quad \begin{vmatrix} 4 & -2 \\ 5 & -6 \end{vmatrix} = -14$, $\quad \begin{vmatrix} 5 & -2 \\ 8 & -6 \end{vmatrix} = -14$)

Proof:



(hint: $A_1 = A_2$)

② $\mathrm{Area}(v_1,v_2) = -\mathrm{Area}(v_2,v_1)$, ie. $\boxed{\text{skew symmetry (anti-symmetry)}}$ (by def

$\det(v_1,v_2) = -\det(v_2,v_1)$

In general, $\det(v_1, v_2, v_i, v_{i+1}, \dots, v_n) = -\det(\dots v_{i+1}, v_i, \dots, v_n)$

$v_i \in \mathbb{R}^n$

Lemma: (determinant formula in $\mathbb{R}^2$)

$$\boxed{\text{Let } f: \mathbb{R}^2 \xrightarrow{\text{Linear}} \mathbb{R}^2 \text{ be given by } A_f = \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \text{ Then } \det(A_f) = ad - bc}$$

Example:

If $A_f = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, then

$$\det(A_f) = \begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} = 3(-2) - (-1)(4) = -2$$

Proof:

$$\text{Area}(\underbrace{a\cdot(1,0) + b(0,1)}_{(a,b)}, \underbrace{c(1,0) + d(0,1)}_{(c,d)}) = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

From ① Bilinearity:

$$\text{Area}((\underset{v_1}{a,0}), (\underset{w}{c,d})) + \text{Area}((\underset{v_2}{0,b}), (\underset{w}{c,d}))$$

$$= \cancel{\text{Area}((a,0),(c,0))} + \text{Area}((a,0),(0,d))$$

$0 \because //$

$$+ \text{Area}((0,b),(c,0)) + \cancel{\text{Area}((0,b),(0,d))}$$

$0 \because //$

$$= \text{Area}((a,0),(0,d)) + \text{Area}((0,b),(c,0))$$

$$= ad - cd$$

② skew-symmetry

Lemma (determinant formula for $\mathbb{R}^3$)

$$\boxed{f: \mathbb{R}^3 \xrightarrow{\text{Linear}} \mathbb{R}^3 \text{ with } A_f = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}. \text{ Then the determinant is } \det(A_f) = aei + gbf + cdh - (ceg + afh + bdi)}$$