The determinant (general formula & how to find l.i.)

→ previous lecture: defined det(f) & compute for 2x2 matrix, formula for 3x3

For 2x2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

For 3x3: $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = (aei + cdh + bfg) - (ceg + bdi + afh)$

What's a formula for det(f) for nxn? → $f: \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} 3 & 7 & -1 & \pi \\ 0 & 6 & -2 & e \\ 1 & -7 & 3 & \ln 2 \\ 4 & 8 & 4 & \cos 3 \end{pmatrix} \leftarrow 4\times4 \text{ matrix}$$

Algorithm:

① Choose either a row or a column

② Develop that row (column) in ①

↳ resulting no. independent of ① and = det(f)

Example

$$\begin{array}{l} \\ \rightarrow \\ \\ \\ \end{array}\begin{pmatrix} 3 & 7 & -1 & \pi \\ 0 & 6 & -2 & e \\ 1 & -7 & 3 & \ln 2 \\ 4 & 8 & 4 & \cos 3 \end{pmatrix} \text{ choice}$$

① $\begin{pmatrix} 3 & 7 & -1 & \pi \\ 0 & 6 & -2 & e \\ 1 & -7 & 3 & \ln 2 \\ 4 & 8 & 4 & \cos 3 \end{pmatrix}$ ② $\begin{pmatrix} 3 & 7 & -1 & \pi \\ 0 & 6 & -2 & e \\ 1 & -7 & 3 & \ln 2 \\ 4 & 8 & 4 & \cos 3 \end{pmatrix}$ compute det, then multiply with 0

Theorem (recursive formula for det)

Let A be an nxn matrix. Given an choice of fixed $i \in \mathbb{N}$, $1 \le i \le n$, then

(fix row i) fix column i

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{ij} \cdot \det(\hat{A}_{ij}) = \sum_{k=1}^{n} (-1)^{i+k} A_{ki} \det(\hat{A}_{ki}),$$ where $\hat{A}_{ij}$ is the $(n-1)\times(n-1)$ submatrix of A given by moving the i-th row & jth column

Remark: geometically, this computing volume / area via projections to low-dim subspace

Example ①,

$\begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix} = 10 + 12 = 22$

$\begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix} \overset{\text{develop 1st row}}{=} 2\cdot(5) + (-4)\cdot 3 = 22$

$\begin{vmatrix} 2 & -4 \\ 3 & -5 \end{vmatrix} \overset{\text{develop 2nd column}}{=} -(-4\cdot3) + 5\cdot2 = 22$

Example ②

$\begin{vmatrix} 3 & 0 & 2 \\ -2 & 1 & 0 \\ 4 & -1 & 0 \end{vmatrix} \overset{\text{dance}}{=} (0+0+4) - (8+0+0) = -4$

$\begin{vmatrix} 3 & 0 & 2 \\ -2 & 1 & 0 \\ 4 & -1 & 0 \end{vmatrix} \overset{\text{1st row}}{=} (3)\cdot\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} - 0\cdot\begin{vmatrix} -2 & 0 \\ 4 & 0 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ 4 & -1 \end{vmatrix}$

$= 0 - 0 - (2)(2) = -4$

$\begin{vmatrix} 3 & 0 & 2 \\ -2 & 1 & 0 \\ 4 & -1 & 0 \end{vmatrix} \overset{\text{3rd column}}{=} (2)\begin{vmatrix} -2 & 1 \\ 4 & -1 \end{vmatrix} = -4$

Def: diagonal:

Given A, its diagonal are the terms of the form a_{ii}

upper Δ upper Δ

Example: $\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} -4 & 2 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ \pi & 2 & 0 & 0 \\ e & 5 & -1 & 0 \\ e^2 & \pi/2 & 3 & 4 \end{pmatrix}$

lower Δ

Formula:

$a_{ij} = 0$ if $i > j$ $a_{ij} = 0$ if $i < j$

If A is upper triangle or lower triangle

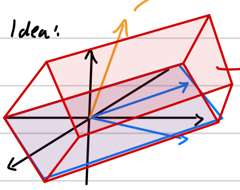
$\boxed{\text{then } \det(A) = \prod_{i=1}^{n} a_{ii}}$ ← product of diagonal terms

Proposition

Let $V = \mathbb{R}^n$, $v_i, \dots, v_n$. Then $v_1, \dots, v_n$ are $\underbrace{\text{linearly dependent}}_{\text{what you want}}$ $\iff \det\begin{pmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & | & | & & | \end{pmatrix} = 0 \rightarrow$ how we check

→ is this linearly independent to → ?

Idea:



→ det is the volume of box,
if →, →, → dependent,
then volume $= 0$

Example:

(i) $\begin{pmatrix} 3 & 2 & 4 & 5 & 8 & 3 \\ -1 & 0 & 0 & 2 & 2 & -1 \\ 3 & 4 & 5 & -7 & 0 & 7 \end{pmatrix}$ 3x6 matrix

Choose any 3x3 submatrix & compute determinant

e.g. $\boxed{\begin{vmatrix} -1 & 0 & 0 \end{vmatrix}} = -6$

Theorem: (really useful)

Let $V = \mathbb{R}^n$, $v_1, \dots, v_k \in V$ $(k \neq n)$

Then $\underbrace{\dim \operatorname{span}(v_1, \dots, v_k)}_{\boxed{\text{maximal no. of linearly independent vectors}}} = \max\{\ell \in \mathbb{N} : \exists\, \ell \times \ell \text{ submatrix of } \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix} \text{ with non-zero det}$