Eigenvalues

$f: V \xrightarrow{\text{Linear}} V$,

goal: ① understand $\ker(f)$, $\mathrm{im}(f)$

(e.g. good basis for f)

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$A_f = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \underset{\substack{\text{basis of} \\ V \text{ s.t.}}}{\overset{\exists \text{ different}}{\sim}} \quad f = N_f = \begin{pmatrix} \text{easy} \\ \text{matrix} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$

diagonal

Key Idea: INTERPRET what it means to be diagonal

We look for directions (lines) s.t. when we apply f, it stays INVARIANT

$f(W) \subseteq W$

THE EQUATION: want $\boxed{v \in V, \text{ s.t. } \exists \lambda \in \mathbb{R} \text{ with } f(v) = \lambda v}$

Lemma:

$$\boxed{\exists v \in V \text{ s.t. } f(v) = \lambda v \iff \underbrace{\det(A_f - \lambda Id)}_{\text{Computable}} = 0}$$

Proof: $f(v) = \lambda v \iff f(v) - \lambda v = 0$

$\underset{\substack{f \text{ given} \\ \text{by } A_f}}{\iff} A_f \cdot v - \lambda \cdot v = 0$

$\iff A_f \cdot v - \boxed{\lambda Id} v = 0$ map

$\iff \boxed{(A_f - \lambda Id)} v = 0$ $\quad v \in \ker(A_f - \lambda Id)$

$\iff \boxed{\det(A_f - \lambda Id) = 0}$ (by dim formula or def of det)

Recall: $\det(B) = 0 \iff$ columns are linearly dependent (as vectors of V)

$B: V \to V$

How do we find possible λ?

↳ $\det(A_f - \lambda \cdot Id) = 0$ is a polynomial equation for $\boxed{\lambda}$ ← solve for λ

Ex 1. $A_f = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$, what are the possible $\boxed{\lambda}$ ← by def called EIGENVALUE

Need to compute,

$(A_f - \lambda \cdot Id) = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$= \begin{pmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{pmatrix}$, so λ is eigenvalue of $\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \iff \begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix} = 0$

$\det \begin{pmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{pmatrix} = (2-\lambda)(-1-\lambda) - 10$

$= -2 - 2\lambda + \lambda + \lambda^2 - 10$

$= \lambda^2 - \lambda - 12$

$= (\lambda - 4)(\lambda + 3) = 0$, $\boxed{\lambda = -3, 4}$ ← these are the eigenvalues

Def:

$\boxed{\text{A vector } v \in V \text{ is said to be an EIGENVECTOR with eigenvalue } \lambda \text{ for } f, \text{ if } f(v) = \lambda v}$

$\boxed{v \in \ker(A_f - \lambda Id)}$

Ex. 1. cont.

What are eigenvectors w/ $\boxed{\lambda = -3}$

Need to compute $\operatorname{Ker}\begin{pmatrix} 5 & 2 \\ 5 & 2 \end{pmatrix}$ $\left(\operatorname{Ker} = \begin{pmatrix} 2-(-3) & 2 \\ 5 & -1-(-3) \end{pmatrix}\right)$

$= \overset{\text{span}}{\langle} (-2, 5) \rangle$

$\Rightarrow (-2, 5)$ is eigenvector of eigenvalue $\lambda = -3$ $(v_{\lambda=-3})$

What are eigenvalues w/ $\lambda = 4$,

Need to compute $\operatorname{Ker}\begin{pmatrix} 2-4 & 2 \\ 5 & -1-4 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix} = \overset{\text{span}}{\langle} (1, 1) \rangle$ $(v_{\lambda=4})$

Given our f, we can write it in the basis $\{v_{\lambda=-3}, v_{\lambda=4}\} = \{(-2,5), (1,1)\}$

$$f = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$$