Diagonalization

$f(v_i) = \lambda_i v_i$

Recall: $f: V \to V \rightsquigarrow \{\lambda_i\}_{1, \dots, \dim V} \rightsquigarrow \{v_i\}$ eigenvector

if λ_i all distinct, then $\{v_i\}$ are a basis

f in the eigenvectors is diagonal

$$f: \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix}$$ eigenvalue

Ex If D is a diagonal matrix: $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$\underbrace{D \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\therefore$ $\boxed{\text{Sol } x_i = \frac{b_i}{\lambda_i} \text{ if } \lambda_i \neq 0 \quad \text{or } x_i \text{ free if } \lambda_i = 0}$ $\Big\}$ dim of space of sol^n is dim ker D = # 0 eigenvalues

The decomposition $A = SDS^{-1}$

eigenvectors

diagonal (easy)

given (complicated)

we had $A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$

$\lambda_1 = -3$, $\lambda_2 = 4 \rightsquigarrow v_{\lambda=-3} = (2, -5)$, $v_{\lambda=4} = (1, 1)$ $\Big\}$ $\boxed{\text{define } S = \begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix}, \text{ then } AS = SD \quad \text{if } \exists S^{-1} \text{ s.t. } SS^{-1} = Id \implies A = SDS^{-1}}$ $\Rightarrow \underset{A}{\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}} \underset{S}{\begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix}} = \underset{S}{\begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}}$

$\therefore \{x \text{ is a sol}^n \text{ to } \overset{\text{hard}}{Ax = b}\} \underset{A = SDS^{-1}}{\Longleftrightarrow} \{y = S^{-1}x \text{ is a sol to } \overset{\text{diagonal (easy system)}}{Dy = S^{-1}b}\}$

$SDS^{-1}x = b$

$D(S^{-1}x) = S^{-1}b$

Ex (cont'd) $\because S = \begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix}$, we want to find $S^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

It must satisfy $\begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \alpha, \beta, \gamma, \delta$ unknown (are unique)

$\Rightarrow \alpha = \frac{1}{7}$

$\beta = \frac{-1}{7}$ (note $\det \begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix} = 7$)

$\gamma = \frac{5}{7}$

$\delta = \frac{2}{7}$

Let's check if $S^{-1} = \begin{pmatrix} \frac{1}{7} & -1/7 \\ 5/7 & 2/7 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$ is the inverse

$S \cdot S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix} \frac{1}{7} \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$

$= \frac{1}{7} \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1/7 & -1/7 \\ 5/7 & 2/7 \end{pmatrix}$

$\Leftrightarrow A = SDS^{-1}$

Theorem (inverse of 2×2 matrix) Let S be 2×2 matrix. Then:

(i) S^{-1} exists $\iff$ $\det(S) \neq 0$ (Pset 4 problem)

in our case always true if S = basis of eigenvectors

(ii) If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$S^{-1} = \frac{1}{\det S}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

How to find S^{-1}

Lemma:

$S^{-1} = \underbrace{\det(S)^{-1}}_{\text{number}} \cdot \underbrace{\operatorname{adj}(S)}_{\text{matrix}}{}^{t}$ → transpose where $\operatorname{adj}(S) = \begin{pmatrix} c_{11} & \cdots & \vdots \\ \vdots & c_{ij} & \vdots \\ \vdots & \cdots & c_{nn} \end{pmatrix}$ has entries $c_{ij} = (-1)^{i+j} \cdot \det\left(c_{ij} \right)$

Ex. $\operatorname{Adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^{t} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$