Applications of $A = SDS^{-1}$

$f: V \xrightarrow{\text{Linear}} V$, A_f = any matrix representation of f (for a basis $\{e_i\}$)

If eigenvalues of f are different, $\exists D$ diagonal, $\exists S$ invertible → eigenvalue $\{v_i\}$

(same as those of A_f) $\begin{pmatrix} \lambda_n & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

s.t. $A = SDS^{-1}$

How can we use $A = SDS^{-1}$

① We want $\det(A)$: we know $\det(XY) \overset{X, Y\ n\times n \text{ matrices}}{=} \det(X)\det(Y)$

If $A = SDS^{-1}$, then $\det(A) = \det(S)\det(D)\det(S^{-1})$

$$= \det(S)\det(D)\frac{1}{\det(S)}$$

$$\boxed{\det(A) = \det(D) = \lambda_1 \cdot \ldots \cdot \lambda_n}$$ product of all eigenvalues

② Taking powers of A: compute $A, A^2, A^3, \ldots, A^{2024}, \ldots, A^{10^{100}}$

Ex. $\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} = A$, what is $A^3 = A \cdot A \cdot A$

$$= A^2 \cdot A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 2 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 32 & 26 \\ 65 & -1 \end{pmatrix}$$ ⤳ complicated

If we use $A = SDS^{-1}$, what is A^2?

It is $A^2 = A \cdot A = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1}$

$$\boxed{\text{Lemma 1: } A^n = SD^nS^{-1}, \text{ note that } D^n = \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{pmatrix}}$$

Ex. $$\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}^{10} = \begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}^{10} \cdot \frac{1}{7}\begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} (-3)^{10} & 0 \\ 0 & (4)^{10} \end{pmatrix}\begin{pmatrix} 1/7 & -1/7 \\ 5/7 & 2/7 \end{pmatrix}$$

$$= \begin{pmatrix} 2(-3)^{10} & (4)^{10} \\ -5(3)^{10} & (4)^{10} \end{pmatrix}\begin{pmatrix} 1/7 & -1/7 \\ 5/7 & 2/7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{7}(-3)^{10} + 5/7(4)^{10} & -2/7(-3)^{10} + 2/7(4)^{10} \\ \frac{-5(3)^{10}}{7} + 5/7(4)^{10} & 5/7(3)^{10} + 2/7(4)^{10} \end{pmatrix}$$ ⟶ easy to compute

③ Computing $\exp(A) = e^A$: recall, if $x \in \mathbb{R}$, then $e^x := \sum_{n \geq 0} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

For a matrix A, define $\underbrace{e^{A}}_{\text{matrix } X}$ $(n\times n)$ $\overset{*}{:=} Id_{n\times n} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots +$

Lemma 2:

$$\boxed{\text{If } A = SDS^{-1}, \text{ then } e^A = S \cdot e^D \cdot S^{-1}}$$

Proof:

$$e^A = e^{SDS^{-1}} \overset{*}{=} Id + SDS^{-1} + \frac{(SDS^{-1})^2}{2!} + \frac{(SDS^{-1})^3}{3!} + \ldots$$

$$\overset{\text{lemma 1}}{=} Id + SDS^{-1} + \frac{SD^2S^{-1}}{2!} + \frac{SD^3S^{-1}}{3!} + \ldots$$

$$= S\left(Id + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \ldots\right)S^{-1}$$

$$= Se^DS^{-1}$$

Application of computing exp of a matrix,

Ex. (22B week 1) $x(t)$, $x: \mathbb{R} \to \mathbb{R}$, we do not know $x(t)$ but we know something about $x'(t)$.

Eg. $x'(t) = 3x$, then $3t + c = x(t)$

$x'(t) = 3x(t) \rightsquigarrow x(t) = e^{3t} \cdot c$

In several variables:

$$\underset{\text{matrix}}{A} \cdot \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} \Longleftrightarrow \begin{pmatrix} a_1 x_1(t) + \cdots + a_n x_n(t) = x_1'(t) \\ \vdots \\ a_{n1} x_1(t) + \cdots + a_{nn} x_n(t) = x_n'(t) \end{pmatrix}$$

$A \cdot x(t) = \vec{x}(t)$ $\quad \therefore \boxed{\text{Sol: } \vec{x} = e^{At}}$

④ Physical meaning of λ_i in quantum mechanics → vector spaces are ∞-dim

In the basic setting, you have Schrodinger's eq^n (analogus of $F = ma$)

$\partial_t \Psi(x,t) = \hat{H} \cdot \Psi(x,t)$, where $H = KE + PE$ and $\boxed{\hat{H}(x,t) = \partial x^2 + V(x)}$ ← linear maps

unknow $C^\infty(\mathbb{R})$ $\qquad \frac{1}{2}(x')^2 \qquad x' = p$

eigenvalues

→ probability for a certain eigenstate