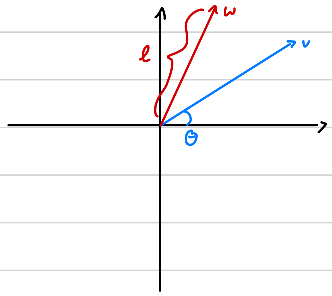
Inner products

$v, w \in V$ vectors:

just with V $\mathbb{R}$-v.s. and linearity,

it does not make sense to discuss/define:



① length $\rightarrow$ $\|v\| := \sqrt{\langle v, v\rangle}$

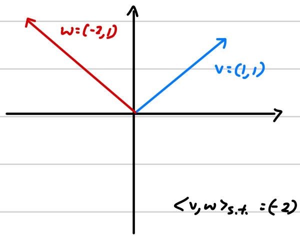
② angles

③ orthogonality ($\perp$) $\rightarrow$ $v \perp w \overset{def}{\iff} \langle v, w\rangle = 0$

Ex. (i) Standard inner product $\langle v, w\rangle_{st}$

defined by $v = (x_1, \dots, x_n)$, $w = (y_1, \dots, y_n)$

$\langle v, w\rangle_{st} := x_1 y_1 + x_2 y_2 + \dots + x_n y_n \in \mathbb{R}$



$\langle v, w\rangle_{s.t.} = (-2)\cdot 1 + 1\cdot 1 = -1$

$$\boxed{\begin{aligned} \langle \cdot, \cdot\rangle : V \times V &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto \langle v, w\rangle \end{aligned}}$$

(ii) Many inner products which are not the standard dot product

For instance, $A = \begin{pmatrix} 10 & 2 \\ 2 & 15 \end{pmatrix}$ gives inner product by

$$\boxed{\langle v, w\rangle_A := \underset{row}{v^t} \cdot A \cdot \underset{col}{w}}$$

$$\therefore \quad \langle v, w\rangle_A = (x_1, x_2) \cdot \begin{pmatrix} 10 & 2 \\ 2 & 15 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 10 x_1 y_1 + \overbrace{\underbrace{2x_1 y_2 + 2x_2 y_1}_{extra}}^{extra} + 15 y_1{}^2$$

E.g. $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle v, w\rangle = 45 + 4 = 49$

What properties we want from an inner product

① First, we want $\langle v, v\rangle$ to be square of length

$\hookrightarrow$

> I. Positivity: $\langle v, v\rangle \geq 0$ for any $v \in V$
>
> II. Definiteness: $\langle v, v\rangle = 0 \Rightarrow v = 0$ (only vector w/o length is 0)

(the only vector w/o length is $\vec{0}$)

② Symmetry: $\boxed{\langle v, w\rangle = \langle w, v\rangle \quad \forall v, w \in V}$

$\rightarrow$ $\boxed{\text{Any } \langle \cdot, \cdot\rangle : V \times V \to \mathbb{R} \text{ bilinear s.t. the properties ① \& ② holds, is said to be an inner product}}$

Lemma

Any $n \times n$ symmetric matrix A gives an inner product $\langle v, w \rangle_A = v^t \cdot A \cdot w$, $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$

$\iff a_{ij} = a_{ji} \ \forall i,j$, $A = A^t$

e.g. $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \neq$ symmetric

$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$ symmetric

iff all eigenvalues of A are +ve

0 not allowed

Ex (i)

Take $A = \begin{pmatrix} -10 & 1 \\ 1 & -25 \end{pmatrix}$, is it positive definite?

Try, $v = (1, 0)$,

$\langle v, v \rangle_A = (1\ 0) \begin{pmatrix} -10 & 1 \\ 1 & -25 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -10$

not positive

Ex (ii)

Take, $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, is it definite?

$(x_1, x_2) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 - 2x_2^2$

so if we choose $v = (1, 1)$, then $\langle v, v \rangle = (1, 1) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

but $v \neq 0$

$\therefore A$ is not definite

From the lemma:

The case of A diagonal is clear by direct computation. Namely if $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ and $\lambda_i \leq 0$,

then $\langle e_i, e_i \rangle_A \leq 0$, so NOT inner product (bc. not positive definite)

In general write $A = SDS^{-1}$, suppose $\exists\, v \in V$ s.t. $v^t D v \leq 0 \iff \exists \lambda_i \leq 0$

then want $\tilde{v}$ s.t. $\tilde{v}^t A \tilde{v} \leq 0$:

Proof:

$\because \tilde{v}^t A \tilde{v} \leq 0 \iff \tilde{v}^t S D S^{-1} \tilde{v} \leq 0$

$\iff (S^t \cdot \tilde{v})^t D (S^{-1} \tilde{v}) \leq 0$

$\boxed{\text{Suppose } S^{-1} = S^t}$, Choose $\tilde{v} = Sv$

Grand-Schmidt

Ex. Decide which of the following give inner products

① $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ ② $\begin{pmatrix} -1 & 3 & -3 \\ 3 & -1 & 3 \\ -3 & 3 & -1 \end{pmatrix}$ ③ $\begin{pmatrix} 4 & 5 & -1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

$\lambda = 1, 3, 4$ $\lambda = -7, 2, 2$ $\lambda = 4, 2, 0, 5$

(ii) Compute lengths & decide orthogonality, (for $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$)

$v = (1, 0, 0)$, its length is $\sqrt{\langle v, v \rangle} = \sqrt{v^t A v}$

$\boxed{\|v\| = 3}$ $= \sqrt{3}$