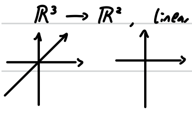
Review for Midterm II

$V \leftarrow \mathbb{R}\text{-v.s.} \xrightarrow{\text{Linear}} W \leftarrow \mathbb{R}\text{-v.s.}$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$, linear systems of eq^n are generalized by



$\rightarrow$ given $w \in W$ find all $v \in V$ s.t. $f(v) = w$

Subspaces: $\operatorname{Ker} f \subseteq V$ and $\operatorname{im} f \subseteq W$

↳ find dimensions of $\operatorname{Ker} f$ & $\operatorname{im} f$

$\rightarrow$ $\boxed{\text{dimension formula: } \dim V = \dim \operatorname{Ker} f + \dim \operatorname{im} f}$

$\rightarrow$ find basis of $\operatorname{Ker} f$ & $\operatorname{im} f$ $\rightarrow$ columns of a matrix form for f

solve $f(v_i) = 0$
& $\{v_1, \dots, v_{\dim \operatorname{Ker} f}\}$
l.i.

From f, we get A_f a matrix (once we choose basis of V & W)

$A_f = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 5 & 7 \end{pmatrix} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, to find basis for $\operatorname{im} f$, scan left $\rightarrow$ right and pick as many columns as you need to get a basis

$\dim \operatorname{im} f = 2$ $\leftarrow$ $\exists$ non-zero 2x2 det

$\dim \operatorname{Ker} f = 1$

$\operatorname{Ker} f$ basis: $(1, 3, -\frac{17}{7})$

$\operatorname{im} f$: $v_1 = (3, 2)$, $v_2 = (-1, 5)$

$\rightarrow$ Eigenvalues & eigenvalues

Purpose: find most simplest matrix expansion of f: diagonal is happiness

$V = \mathbb{R}^3$, $A_f = \begin{pmatrix} 3 & 2 & 0 \\ 1 & -1 & 5 \\ 2 & 3 & -5 \end{pmatrix} = \underset{\text{eigenvectors}}{S} \cdot \overset{\text{diagonal}}{D} \cdot S^{-1}$

① Eigenvalues (λ)

$$\boxed{\left\{ \begin{matrix} f(v) = \lambda v \\ \text{for some } v \end{matrix} \right\} \iff \lambda \text{ is a root of } \det(A_f - \lambda \mathrm{Id}) \iff \det(A_f - \lambda \cdot \mathrm{Id}) = 0}$$

Ex. Is $\lambda = 3$ eigenvalue of A_f?

$$\det(A_f - \underbrace{3 \mathrm{Id}}_{\ominus\, 3 \text{ from each diagonal}}) = \begin{vmatrix} 3 & 2 & 0 \\ 1 & -1 & 5 \\ 2 & 3 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 0 \\ 1 & -4 & 5 \\ 2 & 3 & -8 \end{vmatrix} = (-2)(-18) \neq 0 \quad (\text{Ans: no})$$

Is $\lambda = 0$ eigenvalue? ($\det(A_f - 0 \cdot \mathrm{Id}) = 0$)

$$\det A_f = \begin{vmatrix} 3 & 2 & 0 \\ 1 & -1 & 5 \\ 2 & 3 & -5 \end{vmatrix} = (3)(-10) - (2)(-15) = 0 \quad (\text{Ans: yes})$$

↳ how to find eigenvector? find $\operatorname{Ker}(A_f - \lambda \mathrm{Id})$

for $\lambda = 0$, this is e.g. $v_{\lambda=0} = (2, -3, 1)$

$\rightarrow$ Inner product & norms : $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$

In practice: $\boxed{\text{symmetric positive definite matrix}}$

$A = \begin{pmatrix} 10 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 3 \end{pmatrix}$

so that $\langle v, w \rangle_A = v^t A w = \underset{\text{e.g.}}{(1,0,0)} \begin{pmatrix} 10 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = (10, 2, -1) \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = 28$

↳ Given A square, does it define inner product?

① Symmetric (check)
② Positive definite
- all eigenvalues > 0
- all principal minors > 0

↳ decide orthogonality

$\boxed{v \perp w \iff \langle v, w \rangle = 0}$

are $v = (1, 0, 0)$, $w = (0, 1, 0)$ $\perp$?

no, $\langle v, w \rangle = 2$

but $(1, 0, 0) \perp (0, 1, 2)$

$A_f = \begin{pmatrix} 10 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 3 \end{pmatrix}$

↳ compute lengths

$\boxed{\|v\| := \sqrt{\langle v, v \rangle}}$

Applications & uses:

(i) $A = SDS^{-1}$ useful to compute A^{1000}, e^A

(ii) Decide if A invertible, (and if yes compute A^{-1}) practice 2x2 case

$\dim \operatorname{Ker} f = 0$

(iii) $f: V \xrightarrow{\text{linear}} W$ is injective $\Leftrightarrow$ $\boxed{\operatorname{Ker} f = \{0\}}$ $\left(\overset{\text{if } W = V}{\Leftrightarrow} \det f \neq 0 \right)$

(iv) $f: V \xrightarrow{\text{linear}} W$ is surjective $\Leftrightarrow$ $\boxed{\dim \operatorname{im} f = \dim W}$