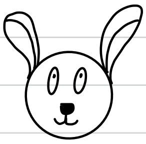
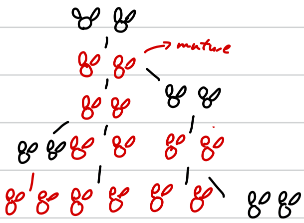


Follow the white rabbit,



in math $n \in \mathbb{N}$,

how many pairs of rabbits

Let F_n = # pairs of rabbits in month n,

Claim: $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$

Kelper's observation:

Look at ratio $\frac{F_{n+1}}{F_n} = \phi$

Ratios:

$\frac{1}{1} = 1, \frac{2}{1} = 2, \frac{3}{2} = 1.5, \frac{5}{3} = 1.\hat{6}, \frac{8}{5} = 1.6, \frac{13}{8} = 1.625, \frac{21}{13} = 1.615, \frac{34}{21} = 1.619, \frac{55}{34} = 1.617, \frac{89}{55} = \boxed{1.618}$

Define $\boxed{\phi = \frac{1+\sqrt{5}}{2} \simeq 1.618\ldots}$ ← golden ratio

Let's try to phrase it in linear algebra terms:

Call $x_n = F_n$, then $\begin{cases} x_{n+1} = x_n + y_n \\ y_{n+1} = x_n \end{cases}$ That is $\boxed{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}}$

$y_n = F_{n-1}$

month n+1 month n

Example:

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\boxed{\begin{pmatrix} x_{16} \\ y_{16} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{15} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$? need to compute A^n

Let's find $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = SDS^{-1}$

→ char, polynomial, $\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 1$

eigenvalues: $\boxed{\lambda_1 = \frac{1+\sqrt{5}}{2}}$, $\boxed{\lambda_2 = \frac{1-\sqrt{5}}{2}}$ → ugly cousin

↳ golden ratio

→ eigenvectors:

for λ_1, $v_{\lambda_1} \in Ker \begin{pmatrix} 1-\lambda_1 & 1 \\ 0 & -\lambda_1 \end{pmatrix}$

for us, $\begin{pmatrix} 1 - \left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix}$

try $v_{\lambda_1} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\lambda_1^2 + \lambda_1 + 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ∵ $\lambda^2 - \lambda - 1 = 0$ for λ_1 & λ_2

Now $A = SDS^{-1}$, $D = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$

$S^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$

$A^n = SD^nS^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$

$A^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2} \cdot \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1^n \\ -\lambda_2^n \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}$

$\boxed{A^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}}$

Theorem:

If $\phi = \frac{1+\sqrt{5}}{2}$, $\mathit{l} = \frac{1-\sqrt{5}}{2}$, then

$F_n = \frac{\phi^n - \mathit{l}^n}{\phi - \mathit{l}} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$

Cor

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2}$$

Proof: Apply our closed formula

$$\frac{F_{n+1}}{F_n} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} \quad \frac{2^n\sqrt{5}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} = \frac{1}{2} \cdot \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n \cdot \left(\frac{1+\sqrt{5}}{2}\right)$$

$$\lim_{n\to\infty} \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n = 0$$