Review & wrap-up

1. V set. Definition of vector space

$(\mathbb{R}^n, \underset{\text{componentwise addition}}{+}, \underset{\text{scalar mult. comp}}{\cdot})$ ✓, $\mathbb{R}[x]$ ✓, V, W s.t. $\mathcal{L}(V,W) = \{f: V \to W : f \text{ linear}\}$ ✓

$\{(x_1, x_2, x_3) : 3x_1 - 4x_2 + 5x_3 = 7\}$ ✗ non-homogeneous, $\{p \in \mathbb{R}[x] : p'(1) = -1\}$ ✗

② Linear maps:

V, W $\mathbb{R}$-v.s., decide if $f: V \to W$ is linear

(i) if $\dim V, \dim W < \infty$, then a choice of basis (V & W) expresses f as a matrix

(ii) $f: \mathbb{R}[x] \to \mathbb{R}[x]$,

$p(x) \longmapsto f(p(x)) := p(-x)$ ← is a linear map

eg. $1 - x + x^3 \overset{f}{\longmapsto} 1 + x - x^3$

conditions: ① $f(p(x) + q(x)) = f(p(x)) + f(q(x))$

② $f(a\,p(x)) = a f(p(x))$

③ Subspaces of a v.s.: $W \subseteq V$ s.t. W itself a v.s. (with operations from V)

Important cases: $\boxed{\ker f \subseteq V, \ \operatorname{im} f \subseteq U}$ if $f: V \to U$ is linear

2 important operations:

(a) Sum, $U_1, U_2 \subseteq V$ then $\boxed{U_1 + U_2 \subseteq V \text{ is a subspace}}$

(b) Intersection: $U_1, U_2 \subseteq V$ then $\boxed{U_1 \cap U_2 \subseteq V \text{ is a subspace}}$

(c) $\boxed{V = U_1 \oplus U_2} \iff V = U_1 + U_2,\ U_1 \cap U_2 = \{0\}$

Ex: $f: \mathbb{R}[x] \to \mathbb{R}[x]$, $p(x) \longmapsto f(p(x)) := p(-x)$ then $\ker f \subseteq \mathbb{R}[x]$

For this $p(-x) = 0$ so $p \equiv 0$ thus $\ker f = \{0\}$ ($\iff f$ injective)

$\operatorname{im} f = V$ bc for any $p(x) \in \mathbb{R}[x]$, $f(p(-x)) = p(x)$ ($\iff f$ surjective)

④ Eigenvalues & eigenvectors of a linear map $f: V \to V$

$\lambda \in \mathbb{R}$ s.t. $\exists v \in V$ with $\boxed{f(v) = \lambda v}$ ← invariant lines of f

In practice, we use $\boxed{\det}$ in finite dimension:

find λ by solving for roots for $\boxed{\text{char. poly}}$. → $\det(f - \lambda \mathrm{Id})$

⊛ in $\dim = \infty$ det ain't a thing

decomposition $A = SDS^{-1}$: D diagonal (keeps λ_i)

↳ compute A^n, e^A (MAT 145) S eigenvect. in columns

↳ discussed invertibility ⟶ how to find inverse matrix? 2×2, 3×3

↳ f invert. $\iff \det f \neq 0$

$\iff \ker = \{0\}$

⑤ Inner product

Ex. cont.

eigenvalues

$f(p(x)) = \lambda \cdot p(x)$,

choose $p(x) = 1 - x^2 + 4x^4 - 7x^6$, then $f(p(x)) = p(x)$ } even powers give $\lambda = 1$, basis $= \{1, x^2, x^4, x^6, \dots\} = U_1$

choose just odd powers, then $f(p(x)) = -p(x)$ } $\lambda = -1$, basis $= \{x, x^3, x^5, x^7, \dots\} = U_2$

$\therefore V = \underset{\text{even}}{U_1} \oplus \overset{\text{odd}}{U_2}$, $\lambda = 1, -1$ ie. $\boxed{f(x) = \underbrace{\left(\frac{f(x) - f(-x)}{2}\right)}_{\substack{\text{odd} \\ U_{\lambda = -1}}} + \underbrace{\left(\frac{f(x) + f(-x)}{2}\right)}_{\substack{\text{even} \\ U_{\lambda = 1}}}}$