University of California Davis **Name (Print):** ____________________

Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Midterm Examination **April 26 2024**

Time Limit: 50 Minutes

This examination document contains 8 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^4$ and consider the two subsets

$$U_1 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0, x_1 - x_3 = 0\},$$

$$U_2 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - x_2 + 2x_3 - 16x_4 = 0, x_4 = 0\}.$$

   (a) (5 points) Show that $U_1 \subseteq V$ is a vector subspace.

   (b) (10 points) Show that $V = U_1 \oplus U_2$.

(c) (5 points) Show that $\{v_1, v_2\}$ is a basis of U_1 where

$$v_1 = (1, -1, 1, -1), \quad v_2 = (1, -2, 1, 0).$$

(d) (5 points) Find a basis for U_2.

2. (25 points) Consider the vector space $V = \mathbb{R}^3$ and the vectors

$$v_1 = (1, 0, -1), \quad v_2 = (3, 2, -1), \quad v_3 = (9, 4, -5),$$

$$v_4 = (0, 1, 0), \quad v_5 = (1, 1, 1).$$

(a) (5 points) Show that $V \neq \operatorname{span}(v_1, v_2, v_3)$.

(b) (10 points) Consider the subspace $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_3 = 0\}$. Prove that $U = \operatorname{span}(v_4, v_5)$.

(c) (5 points) Prove that $\{v_1, v_4, v_5\}$ is a basis of V.

(d) (5 points) Find the dimension of the intersection $\text{span}(v_1, v_2, v_3) \cap \text{span}(v_4, v_5)$.

3. (25 points) Let $V = \mathbb{R}[x]$ and solve the following parts.

   (a) (15 points) Consider the function $f : V \to \mathbb{R}$ given by $f(p(x)) = p(4)$. Show that f is a linear function.

   (b) (10 points) Find an infinite set of linearly independent vectors $\{v_1, v_2, \ldots, v_i, \ldots\}$, $v_i \in V$, such that $f(v_i) = 0$.

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) The function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by

   $$f(x_1, x_2, x_3) = (x_1, x_2 + 1, 3x_1 - x_2, x_1 + x_2)$$

   is a linear function.

   (1) True. (2) False.

   (b) (5 points) Let $U_1, U_2, U_3 \subseteq V$ be three vector subspaces of a vector space V such that $U_1 \oplus U_2 = U_1 \oplus U_3$. Then $U_2 = U_3$.

   (1) True. (2) False.

   (c) (5 points) Let $\{v_1, \dots, v_n\}$ be a set of vectors in V such that $w \in \text{span}(v_1, \dots, v_n)$ for any $w \in V$. Then $\{v_1, \dots, v_n\}$ is a basis.

   (1) True. (2) False.

   (d) (5 points) Let $V = \mathbb{R}^4$ and $U_1, U_2 \subseteq V$ be two 2-dimensional planes. Then their intersection $U_1 \cap U_2$ must be a line in V.

   (1) True. (2) False.

   (e) (5 points) Let $V = \mathbb{R}[x]$. Then $U = \{p(x) \in V : p''(1) = 0\}$ is a vector subspace.

   (1) True. (2) False.

Blank page for computations.