University of California Davis **Name (Print):** ____________________

Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Midterm Examination II **May 31 2024**

Time Limit: 50 Minutes

This examination document contains 10 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^3$ and consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A_f = \begin{pmatrix} 2 & -1 & -1 \\ -4 & 4 & 4 \\ -3 & 4 & 4 \end{pmatrix}$$

in the standard coordinate basis.

(a) (7 points) Show that the eigenvalues of A_f are $\lambda = 0, 1, 9$.

(b) (8 points) Find a basis of eigenvectors for A_f.

(c) (10 points) Write $A_f = S \cdot D \cdot S^{-1}$ where D is a diagonal matrix.

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2. (25 points) Let $V = \mathbb{R}^3$ and consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A_f = \begin{pmatrix} 5 & 5 & 5 \\ 2 & 0 & 3 \\ 3 & 5 & 2 \end{pmatrix}$$

in the standard coordinate basis.

(a) (5 points) Show that $\dim(\ker(f)) = 1$.

(b) (10 points) Find a basis of the kernel $\ker(f)$.

(c) (5 points) Find a basis of the image $\mathrm{im}(f)$.

(d) (5 points) Prove or disprove whether f is invertible.

3. (25 points) Consider the vector space $V = \mathbb{R}^2$ and consider the $\mathbb{R}$-bilinear binary operation $\langle \cdot, \cdot \rangle_A : V \times V \to \mathbb{R}$ given by

$$\langle v, w \rangle_f = v^t \cdot A \cdot w, \quad \text{where } A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}.$$

(a) (10 points) Show that $\langle \cdot, \cdot \rangle_A$ is an inner product.

(b) (6 points) Compute the lengths of the vectors $(1, 0)$ and $(1, 1)$.

(c) (5 points) Are $(1, 0)$ and $(0, 1)$ orthogonal with respect to $\langle \cdot, \cdot \rangle_A$?

(d) (4 points) Find a basis $\{v_1, v_2\}$ of $\mathbb{R}^2$ of orthogonal vectors with respect to $\langle \cdot, \cdot \rangle_A$.

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

(a) (5 points) The eigenvalues of $\begin{pmatrix} 1 & -4 \\ 0 & 7 \end{pmatrix}$ are $\lambda = -4, 7$.

(1) True. (2) False.

(b) (5 points) If $f : \mathbb{R}^{22} \to \mathbb{R}^{15}$ has $\dim(\ker f) = 7$, then $\dim(\mathrm{im} f) = 8$.

(1) True. (2) False.

(c) (5 points) The vectors $(1, 0, 2, 0), (3, 4, 0, 0)$, $(0, 5, 1, -1)$ are linearly independent.

(1) True. (2) False.

(d) (5 points) The inverse of $A = \begin{pmatrix} 2 & 7 \\ 3 & 11 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 11 & -7 \\ -3 & 2 \end{pmatrix}$.

(1) True. (2) False.

(e) (5 points) Let $V = \mathbb{R}[x]$. The map $f : V \to V$ given by $f(p(x)) = p'(x)$ is injective.

(1) True. (2) False.

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