University of California Davis **Name (Print):** ____________________

Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Practice Final I **June 13 2024**

Time Limit: 120 Minutes

This examination document contains 8 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^4$ and consider the subspace

$$U = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 - 5x_4 = 0, \quad x_1 + 3x_2 + 4x_4 = 0\},$$

   (a) (7 points) Show that $U \subseteq V$ is a vector subspace of V and compute its dimension.

   (b) (10 points) Find a subspace $W \subseteq V$ such that $V = U \oplus W$.

(c) (8 points) Find a basis for the intersection $U \cap \ker f$, where $f : \mathbb{R}^4 \to \mathbb{R}^2$ is the linear map given (in the standard basis) by the matrix

$$A_f = \begin{pmatrix} 0 & 4 & -4 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

(d) (10 points) Show that the map $f : \mathbb{R}^4 \to \mathbb{R}^2$ in (c) is surjective.

2. (25 points) Let $V = \mathbb{R}^3$ and consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A_f = \begin{pmatrix} 4 & -3 & 5 \\ -4 & 5 & -5 \\ 5 & -5 & 3 \end{pmatrix}$$

in the standard coordinate basis.

(a) (5 points) Show that $v = (11, -13, 12)$ is an eigenvector with eigenvalue $\lambda = 13$.

(b) (10 points) Find S invertible and D diagonal such that $A_f = S \cdot D \cdot S^{-1}$.

(c) (5 points) Consider the parallelepiped P in $\mathbb{R}^3$ spanned by the vectors

$$v_1 = (4, -4, 5), \quad v_2 = (-3, 5, -5), \quad v_3 = (5, -5, 3).$$

Show that the absolute value of its volume is $|\mathrm{vol}(P)| = 26$.

(d) (5 points) Show that A_f is invertible and its inverse A_f^{-1} is given by the matrix

$$A_f^{-1} = \frac{1}{26} \cdot \begin{pmatrix} 10 & 16 & 10 \\ 13 & 13 & 0 \\ 5 & -5 & -8 \end{pmatrix}.$$

3. (25 points) Let $V = \mathbb{R}^3$ endowed with bilinear operation defined by

$$\langle v, w \rangle_A = v^t \cdot A \cdot w, \quad \text{where } A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

(a) (8 points) Show that $\langle \cdot, \cdot \rangle_A$ is an inner product.

(b) (10 points) Consider the subspace $U \subseteq V$ given by

$$U = \{(x_1, x_2, x_3) : x_1 - 4x_2 - 5x_3 = 0\} \subseteq V.$$

Show that $v_1 = (5, 0, 1)$ and $v_2 = (0, 15, -12)$ are an orthogonal basis[1] of U.

[1]That is, prove that v_1, v_2 are a basis of U *and* that v_1, v_2 are orthogonal with respect to $\langle \cdot, \cdot \rangle_A$.

(c) (3 points) Compute the norms of v_1 and v_2 with respect to $\langle \cdot, \cdot \rangle_A$.

(d) (4 points) Find a vector $v \in V$ such that $V = U \oplus \text{span}(v)$ and the norm of v is 1.

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) The map $f : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $f(p(x)) := p''(6) + 2$ is a linear map.

   (1) True. (2) False.

   (b) (5 points) Let V be a vector space of $\dim(V) = 8$ and $f : V \to V$ a linear map. Then $\det(2 \cdot f) = 2 \cdot \det(f)$.

   (1) True. (2) False.

   (c) (5 points) Any linear map $f : \mathbb{R}^{10} \to \mathbb{R}^6$ is surjective.

   (1) True. (2) False.

   (d) (5 points) A linear map $f : \mathbb{R}^{10} \to \mathbb{R}^6$ cannot be injective.

   (1) True. (2) False.

   (e) (5 points) If $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, then $A^7 = 5^6 \cdot A$.

   (1) True. (2) False.