University of California Davis **Name (Print):** ____________________
Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Practice Final II **June 13 2024**
Time Limit: 120 Minutes

This examination document contains 8 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^4$ and consider the linear map $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$A_f = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}.$$

Consider also the subset

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - x_2 + 3x_3 + 2x_4 = 0\} \subseteq V$$

(a) (7 points) Show that $U \subseteq V$ is a vector subspace of V.

(b) (8 points) Find the dimensions of $\dim(\ker f)$ and $\dim(\operatorname{im} f)$.

(c) (10 points) Show that $V = U + \ker f$.

(d) (10 points) Prove or disprove whether $V = U \oplus \ker f$.

2. (25 points) Let $V = \mathbb{R}^3$ and consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A_f = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

in the standard coordinate basis.

(a) (5 points) Show that $\lambda = 5$ is an eigenvalue of A_f and find the other two eigenvalues.

(b) (10 points) Find a basis of eigenvectors of A_f.

(c) (5 points) Show that A_f defines an inner product on $V = \mathbb{R}^3$ via

$$\langle v, w \rangle = v^t \cdot A_f \cdot w.$$

(d) (5 points) Prove or disprove whether $v = (1, 1, 1)$ and $w = (0, -1, 1)$ are orthogonal with respect to the inner product defined by A_f.

3. (25 points) Consider the vector space $V = \mathbb{R}[x]$ and consider the map $f : \mathbb{R}[x] \to \mathbb{R}[x]$ given by

$$f(p(x)) = p_7(x),$$

where $p_7(x)$ is the polynomial consisting of terms of $p(x)$ of degree at most 7. For instance, $f(1 + x^4 - 5x^7 + x^9 - 14x^{11}) = 1 + x^4 - 5x^7$.

   (a) (10 points) Show that f is a linear map.

   (b) (6 points) Find a basis of $\ker(f)$ and a basis of $\operatorname{im}(f)$.

(c) (5 points) Let $U \subseteq \mathbb{R}[x]$ be the subspace of polynomials whose terms are all even powers, i.e. $U = \{p(x) \in \mathbb{R}[x] : p(x) = p(-x)\}$. Compute the dimension of the intersection $\operatorname{im}(f) \cap U$.

(d) (4 points) Does f have any eigenvalue apart from $\lambda = 0$? (Justify your answer.)

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) The map $f : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $f(p(x)) := \int_0^x p(\xi)d\xi$ is a linear map.

   (1) True. (2) False.

   (b) (5 points) Let v_1, v_2 be eigenvectors of eigenvalues λ_1, λ_2 respectively. If $\lambda_1 \neq \lambda_2$ then v_1, v_2 are linearly independent.

   (1) True. (2) False.

   (c) (5 points) $(1, 0, 1, 0), (-2, -4, 5, 7), (1, -4, 8, 7)$ are linearly independent.

   (1) True. (2) False.

   (d) (5 points) The inverse of $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is A itself.

   (1) True. (2) False.

   (e) (5 points) Let $V = \mathbb{R}[x]$. The map $f : V \to V$ given by $f(p(x)) = p'(x)$ is surjective.

   (1) True. (2) False.