MAT 67: PRACTICE PROBLEMS FOR MIDTERM II

NO DUE DATE

Abstract. These are some practice problems for the 2nd Midterm of MAT-67. These are an addition to the already available two practice midterms for the 2nd Midterm. The 2nd Midterm takes place on Friday May 31 2024.

Computational Problems

Problem 1. Decide whether the following sets of vectors are linearly independent:

(1) Let $v_1, v_2, v_3 \in \mathbb{R}^3$ where
$$v_1 = (1,3,0), \quad v_2 = (3,2,0), \quad v_3 = (5,8,0).$$

(2) Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^5$ where
$$v_1 = (1,0,0,0,0), \quad v_2 = (0,1,0,2,0), \quad v_3 = (0,1,1,0,0), \quad v_4 = (0,0,1,0,1).$$

(3) Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^4$ where
$$v_1 = (1,1,1,1), \quad v_2 = (1,1,-1,1),$$
$$v_3 = (2,0,0,0), \quad v_4 = (0,0,0,1).$$

Problem 2. For each of the following matrices A, show that the eigenvalues are as given, find corresponding eigenvectors, and compute the matrices S, D such that $A = S \cdot D \cdot S^{-1}$.

(1) $A = \begin{pmatrix} -5 & -3 \\ -3 & 3 \end{pmatrix}$ has eigenvalues $\lambda = -6, 4$.

(2) $A = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$ has eigenvalues $\lambda = -1, 9$.

(3) $A = \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}$ has eigenvalues $\lambda = -2, 3$.

(4) $A = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$ has eigenvalues $\lambda = 1, 3$.

(5) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ has eigenvalues $\lambda = -1, 0, 1$.

(6) $A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 1 & -3 \\ 0 & -3 & -2 \end{pmatrix}$ has eigenvalues $\lambda = -4, -2, 3$.

(7) $A = \begin{pmatrix} 1 & -1 & 2 & 2 \\ -2 & -1 & 1 & 2 \\ 2 & 0 & -2 & -2 \\ 2 & 1 & -2 & 1 \end{pmatrix}$ has eigenvalues $\lambda = -5, -1, 2, 3$.

Problem 3. For each of the following matrices A, decide whether they define an inner product via $\langle v, w\rangle_A := v^t A w$.

(1) $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. (*Answer: no, not positive definite.*)

(2) $A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$. (*Answer: yes, symmetric and positive definite.*)

(3) $A = \begin{pmatrix} 2 & -3 & 3 \\ -3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$. (*Answer: no, not positive definite.*)

(4) $A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$. (*Answer: yes, symmetric and positive definite.*)

(5) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}$. (*Answer: yes, symmetric and positive definite.*)

(6) $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & -2 & -1 \end{pmatrix}$. (*Answer: no, not positive definite.*)

Problem 4. For each of the inner products in Problem 3.(2), 3.(4) and 3.(5), compute the lenghts of the vectors $(1, -1)$, $(1, 2, 1)$ and $(4, 0, 2)$ respectively, and find two *orthogonal* non-zero vectors of $V = \mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^3$ respectively. (Orthogonal with respect to those inner products.)

Problem 5. Compute the inverses of the matrices in Problem 2.(1), 2.(2) and 2.(6).

Problem 6. For each of the matrices A in Problem 2.(1), 2.(2) and 2.(6), compute A^{1000} and e^A.

Problem 7. Consider the linear map $f : \mathbb{R}^5 \to \mathbb{R}^5$ given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(1) Show that $\dim(im(f)) = 3$ and $\dim(\ker(f)) = 2$.

(2) Find a basis of $\ker(f)$ and a basis of $im(f)$.

(3) Show that $\lambda = 3$ is an eigenvalue of f and find an eigenvector with eigenvalue $\lambda = 3$.

Problem 8. For each of the matrices A in Problem 3.(1), 3.(4) and 3.(6), find a basis of $\ker(A)$ and $im(A)$.

Conceptual Problems

Problem 1. From the textbook, the four *Proof-Writing Exercises* in Chapter 7.(3), 7.(4), 7.(8) and 7.(9) on eigenvalues and eigenvectors.

Problem 2. From book, *Proof-Writing Exercises* in Chapter 8.(1), 8.(3) and 8.(4) on inner products and norms.

Problem 3. Let $V = \mathbb{R}[x]$ be the space of real polynomials on the variable x. Consider the operation $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ given by

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x) \cdot q(x) dx$$

(1) Show that this operation is bilinear and symmetric.

(2) Show that it is positive definite, and thus it defines an inner product in V.

(3) Compute the norms of the following four vectors:

$$1, \quad 1 + x - x^2, \quad 2 - x^3, \quad x^5 + 10x^{11}.$$

(4) Check whether the following five vectors are mutually orthogonal:

$$1, \quad x, \quad 3x^2 - 1, \quad 35x^4 - 30x^2 + 3, \quad 63x^5 - 70x^3 + 15x.$$

Problem 4. Find linear maps with the following properties or show they do not exist:

(1) An injective map $f : V \to V$ from a finite-dimensional vector space to itself which admits no inverse.

(2) An injective map $f : V \to V$ from a vector space to itself with no inverse.

(3) A non-surjective map $f : \mathbb{R}^{21} \to \mathbb{R}^{17}$ with no kernel.

(4) A surjective map $f : \mathbb{R}^{21} \to \mathbb{R}^{17}$ with no kernel.

(5) A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that there is no basis of $\mathbb{R}^2$ made of eigenvectors of f.

(6) A map $f : \mathbb{R}^{18} \to \mathbb{R}^{18}$ with all eigenvalues equal to $\lambda = 1$ but not equal to the identity.

(7) A map $f : \mathbb{R}^{18} \to \mathbb{R}^{18}$ with all eigenvalues equal to $\lambda = 1$, whose eigenvectors form a basis of $\mathbb{R}^{18}$ and it is not equal to the identity.

(8) A map $f : V \to V$ with all eigenvalues different but such that there is no basis of V made of eigenvectors of f.