LECTURE 1: SOLUTIONS TO PRACTICE EXERCISES

MAT-67 SPRING 2024

Abstract. These are solutions to the practice problems corresponding to the first lecture of MAT-67 Spring 2024, delivered on April 1st 2024. Solutions were typed by TA Scroggin, please contact *tmscroggin – at – ucdavis.edu* for any comments.

Problem 1. For each of the following eight systems of equations, decide whether the system is *linear* or *non-linear*.

(1)

$$\begin{cases} 3x_1 + 2x_2 - 4.7x_3 = 5 \\ -x_1 + 9.1x_2 - 2x_3 = 10 \end{cases}$$

(2)

$$\begin{cases} 3x_1^7 + 2x_2 - x_1x_3 = 0 \\ -x_1 + 9.1x_2 - 2x_3 = 10 \end{cases}$$

(3)

$$\begin{cases} x_1x_2 = 1 \\ x_1 + x_2 = -1 \end{cases}$$

(4)

$$\begin{cases} x_2 + \sqrt{x_3} = 1 \\ x_1 - 2x_3 = 10 \\ \cos(x_2) - x_3 = 0 \end{cases}$$

(5)

$$\begin{cases} e^{x_1}x_2 + 4x_3 + 7x_4 = -16 \\ x_2 - x_3 + x_4 = 10 \\ x_1 + x_2x_4 - 8x_3 = 1 \end{cases}$$

(6)

$$\begin{cases} x_1 - x_2 + 5x_3 - 9x_4 = 1 \\ 3x_1 - 8x_2 + 5x_3 - 9x_4 = 0 \\ -4x_1 + 5x_2 + 5x_3 - \ln(2)x_4 = 1 \\ 5x_1 - x_2 + 10x_3 - 9x_4 = \cos(105) \end{cases}$$

(7)

$$\begin{cases} e^3x_2 + 4x_3 + \sin(54)x_4 = -\ln(\cos(1 + e^7)) \\ \tan(32)x_2 - x_3 + x_4 = 10 \\ x_1 + x_2 - x_4 - 8x_3 = 1 \end{cases}$$

(8)

$$\begin{cases} x_1 + x_2 = -2 \\ (x_1 + x_2)^2 - 2x_1x_2 = 10 \end{cases}$$

Solution. Recall that linear equations must satisfy vector addition ($f(x+y) = f(x) + f(y)$ for vectors x and y) and scalar multiplication ($f(cx) = cf(x)$ for some scalar c).

(1) *Claim*: The system of equations is **linear**.
Both equations are functions of variables with highest degree 1 and are absent of products of variables or special functions (e.g. trigonometric, logarithmic or exponential functions) that satisfy vector addition and scalar multiplication.

(2) *Claim*: The system of equations is **non-linear**.
The first equation $3x_1^7 + 2x_2 - x_1x_3 = 0$ is non-linear since x_1 is degree 7 and there is a product of x_1 and x_3.

(3) *Claim*: The system of equations is **non-linear**.
The first equation $x_1x_2 = 1$ contains a product of variables.

(4) *Claim*: The system of equations is **non-linear**.
The first equation $x_2 + \sqrt{x_3} = 1$ has a variable of degree $1/3$, and the third equation $\cos(x_2) - x_3 = 0$ contains a trigonometric function.

(5) *Claim*: The system of equations is **non-linear**.
The first equation contains an exponential function as well as a product of variables. The third equation contains a product of variables.

(6) *Claim*: The system of equations is **linear**.
All four equations are functions of variables with highest degree 1 and are absent of products of variables or special functions. Please note that $\ln(2)$ and $\cos(105)$ are scalars.

(7) *Claim*: The system of equations is **linear**.
All three equations are functions of variables with highest degree 1 and are absent of products of variables or special functions. Please note that $e^3, \sin(54), -\ln(\cos(1 + e^7)), \tan(32)$ are all scalars.

(8) *Claim*: The system of equations is **non-linear**.
The second equation $(x_1+x_2)^2 - 2x_1x_2 = 10$ simplifies to $x_1^2 + x_2^2 = 10$ which contains variables of degree 2.

□

Problem 2. By direct calculation, discuss whether each of the following linear systems of equations have *no solution*, *a unique solution* or *infinitely many solutions*.

(1) The following linear system in two unknown variables $x_1, x_2 \in \mathbb{R}$:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \end{cases}$$

(2) The following linear system in two unknown variables $x_1, x_2 \in \mathbb{R}$:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 1 \end{cases}$$

(3) The following linear system in three unknown variables $x_1, x_2, x_3 \in \mathbb{R}$:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_3 = 1 \end{cases}$$

(4) The following linear system in three unknown variables $x_1, x_2, x_3 \in \mathbb{R}$:

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + 4x_2 + x_3 = 2 \\ x_1 + x_2 + 5x_3 = -12 \end{cases}$$

(5) The following linear system in three unknown variables $x_1, x_2, x_3 \in \mathbb{R}$:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + 6x_3 = 0 \\ x_1 + x_2 + 5x_3 = 2 \end{cases}$$

(6) The following linear system in three unknown variables $x_1, x_2, x_3 \in \mathbb{R}$:

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + 6x_3 = 0 \\ x_1 + x_2 + 5x_3 = 0 \end{cases}$$

Solution. (1) *Claim*: The linear system of equations has **no solution**.
Subtracting equation (1) from equation (2) results in the equation $0 = 1$, for which there is no solution.

(2) *Claim*: The system of equations has a **unique solution** of $(x_1, x_2) = (\frac{1}{2}, -\frac{1}{2})$.
Adding equation (1) and equation (2) together results in $2x_1 = 1$. Solving for x_1 we get $x_1 = \frac{1}{2}$. Plugging $x_1 = \frac{1}{2}$ into either equation (1) or equation (2) allows us to solve for $x_2 = -\frac{1}{2}$.

(3) *Claim*: The system of equations has **infinitely many solutions** that satisfy $(x_1, x_2, x_3) = (-x_3 + 1, x_3 - 1, x_3)$.
Observe that there are 2 equations and 3 unknowns, this suggests that we cannot completely solve for a unique solution and therefore, we either have infinitely many solutions or no solution.
Subtracting equation (2) from equation (1), we find that $x_2 - x_3 = -1$. Solving for

x_2 we find that $x_2 = x_3 - 1$, now we may solve for x_1 by plugging our solution for x_2 into equation (1) and we find that

$$x_1 + x_2 = 0$$
$$x_1 + (x_3 - 1) = 0$$
$$x_1 = -x_3 + 1$$

Finally, we find that $x_1 = -x_3 + 1, x_2 = x_3 - 1$, and $x_3 = x_3$.
Alternatively, one may have solved for x_3 initially and found the solution $(x_1, x_2, x_3) = (-x_2 - 1, x_2, x_2 + 1)$.

(4) *Claim*: The system of equations has a **unique solution** of $(x_1, x_2, x_3) = (\frac{7}{3}, \frac{2}{3}, -3)$.
First, subtract equation (1) from equation (2).

$$x_1 + 4x_2 + x_3 = 2 \qquad (2)$$
$$-(x_1 + x_2 + x_3 = 0) \qquad -(1)$$

$$3x_2 = 2$$
$$x_2 = \frac{2}{3}$$

Now, subtract equation (1) from equation (3).

$$x_1 + x_2 + 5x_3 = -12 \qquad (3)$$
$$-(x_1 + x_2 + x_3 = 0) \qquad (1)$$

$$4x_3 = -12$$
$$x_3 = -3$$

Finally, we may solve for x_1 by plugging solutions for x_2 and x_3 into either equations (1), (2) or (3). Here, I have chosen equation (1)

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + \frac{2}{3} - 3 = 0$$
$$x_1 - \frac{7}{3} = 0$$
$$x_1 = \frac{7}{3}$$

(5) *Claim*: The linear system of equations has **no solutions**.
First, we subtract equation (1) from equation (3).

$$x_1 + x_2 + 5x_3 = 2 \qquad (3)$$
$$-(x_1 + x_2 + x_3 = 1) \qquad -(1)$$

$$4x_3 = 1$$
$$x_3 = \frac{1}{4}$$

Now, we add -2 times equation (1) to equation (2).

$$\begin{aligned} 2x_1 + 2x_2 + 6x_3 &= 0 && (2) \\ -2(x_1 + x_2 + x_3 &= 1) && -2(1) \\ \hline 4x_3 &= -2 \\ x_3 &= -\frac{1}{2} \end{aligned}$$

We reach a contradiction, since $\frac{1}{4} \neq -\frac{1}{2}$. Therefore, there are no solutions.

(6) *Claim*: The linear system of equations has **infinitely many solutions** of the form $(x_1, x_2, x_3) = (x_1, -x_1, 0)$.
First, we add -2 times equation (1) to equation (2).

$$\begin{aligned} 2x_1 + 2x_2 + 6x_3 &= 0 && (2) \\ -2(x_1 + x_2 + x_3 &= 0) && -2(1) \\ \hline 4x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

Then we add -1 times equation (1) to equation (3).

$$\begin{aligned} x_1 + x_2 + 5x_3 &= 0 && (3) \\ -(x_1 + x_2 + x_3 &= 0) && -(1) \\ \hline 4x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

Therefore, $x_3 = 0$. Now, if we plug $x_3 = 0$ into either equation (1), (2) or (3), we find that the $x_1 + x_2 = 0$. Solving for x_2, we find that $x_2 = -x_1$ and the full set of solutions is $(x_1, x_2, x_3) = (x_1, -x_1, 0)$ as claimed.
Alternatively, we may have solved for x_1 and found the solution $x_1 = -x_2$ and the full set of solutions is $(x_1, x_2, x_3) = (-x_2, x_2, 0)$.

□

Problem 3. For each of the linear systems in Problem 2, write a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ and an m-tuple $(y_1, \ldots, y_m) \in \mathbb{R}^m$, for some $n, m \in \mathbb{R}$, (all depending on the given system) such that solving that given linear system is equivalent to finding $(x_1, \ldots, x_n) \in \mathbb{R}^n$ in the domain of f such that

$$f(x_1, \ldots, x_n) = (y_1, \ldots, y_m).$$

Solution. Note that for the linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the number of variables in the system of equations corresponds to the dimension of the domain, n, whereas the number of equations corresponds to the dimension of the codomain, m.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x_1, x_2) = (x_1 + x_2, x_1 + x_2)$ given that $y = (0, 1)$.
(2) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ given that $y = (0, 1)$.
(3) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_3)$ given that $y = (0, 1)$.
(4) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ where $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 + 4x_2 + x_3, x_1 + x_2 + 5x_3)$ given that $y = (0, 2, -12)$.
(5) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ where $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + 2x_2 + 6x_3, x_1 + x_2 + x_3)$ given that $y = (1, 0, 2)$.
(6) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ where $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + 2x_2 + 6x_3, x_1 + x_2 + 5x_3)$ given that $y = (0, 0, 0)$.

□