LECTURE 2: PRACTICE EXERCISES

MAT-67 SPRING 2024

Abstract. These practice problems correspond to the 2nd lecture of MAT-67 Spring 2024, delivered on April 3rd 2024. Solutions were typed by TA Scroggin, please contact *tmscroggin – at – ucdavis.edu* for any comments.

Recall that a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be linear if it satisfies the following 2 conditions:

(i) $f(x+y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}^n$,
(ii) $f(c \cdot x) = c \cdot f(x)$, for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

See lecture notes from Lectures 1 & 2, and also Section 1.3 in book, for more details.

Problem 1. For each of the following maps, prove whether it is *linear* or *non-linear*.

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 5x$,
(2) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 5x + 1$,
(3) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \cos(x)$,
(4) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3 - x$,
(5) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \ln(1 + x^2)$,
(6) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x_1, x_2) = x_1 + 4x_2$,
(7) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x_1, x_2) = 3x_1 - x_2 + 7$,
(8) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (3x_1 - x_2, x_2)$,
(9) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, 4x_1 + x_3)$,
(10) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, 1)$
(11) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1x_3, x_1 - x_2)$
(12) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, $f(x_1, x_2, x_3) = (e^{x_3 + x_1}, 3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, 0)$

Solution. Please note that I shall use the distributive law below without explicitly mentioning this fact, due to the number of exercises. However, in your proof you should state when this rule is applied.

(1) *Claim*: The function is **linear**.
proof: We verify that f satisfies conditions (i) and (ii):
(i) By the distributive law, $f(x+y) = 5(x+y) = 5x + 5y = f(x) + f(y)$,
(ii) $f(cx) = 5(cx) = 5cx = c(5x) = cf(x)$.

(2) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii):
(i) Given that

$$f(x+y) = 5(x+y) + 1 = 5x + 5y + 1,$$
$$f(x) + f(y) = (5x+1) + (5y+1) = 5x + 5y + 2$$

then $f(x+y) \neq f(x) + f(y)$.
(ii) $f(cx) = 5(cx) + 1 = c(5x) + 1 \neq c(5x) + c = c(5x+1) = cf(x)$.

(3) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii):
(i) $f(x+y) = \cos(x+y) \neq \cos(x) + \cos(y) = f(x) + f(y)$,
(ii) $f(c \cdot x) = \cos(cx) \neq c \cdot \cos(x) = c \cdot f(x)$.

(4) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii):
(i) $f(x+y) = (x+y)^3 + (x+y) = x^3 + 3x^2y + 3xy^+y^3 + x + y \neq x^3 + y^3 + x + y = f(x) + f(y)$,
(ii) $f(c \cdot x) = (cx)^3 + cx = c^3x^3 + cx = c(c^2x^3 + x) \neq c(x^3 + x) = c \cdot f(x)$.

(5) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii).
(i) $f(x+y) = \ln(1+(x+y)^2) = \ln(1+x^2+2xy+y^2) \neq \ln(1+x^2) + \ln(1+y^2) = f(x) + f(y)$,
(ii) $f(c \cdot x) = \ln(1+(cx)^2) = \ln(1+c^2x^2) \neq \ln(1+x^2)^c = c\ln(1+x^2) = c \cdot f(x)$.

(6) *Claim*: The function is **linear**.
proof: We verify that f satisfies conditions (i) and (ii):
(i)

$$\begin{aligned} f((x_1,x_2)+(y_1,y_2)) &= f(x_1+y_1, x_2+y_2) = (x_1+y_1) + 4(x_2+y_2) \\ &= x_1 + y_1 + 4x_2 + 4y_2 = (x_1 + 4x_2) + (y_1 + 4y_2) \\ &= f(x_1,x_2) + f(y_1,y_2), \end{aligned}$$

(ii) $f(c \cdot (x_1,x_2)) = f(cx_1, cx_2) = (cx_1) + 4(cx_2) = c(x_1 + 4x_2) = c \cdot f(x_1,x_2)$.

(7) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii).
(i) Given that

$$\begin{aligned} f(x_1+y_1, x_2+y_2) &= 3(x_1+y_1) - (x_2+y_2) + 7 \\ &= 3x_1 + x_2 + 3y_1 + y_2 + 7 \\ f(x_1,x_2) + f(y_1,y_2) &= (3x_1 + x_2 + 7) + (3y_1 + y_2 + 7) \\ &= 3x_1 + x_2 + 3y_1 + y_2 + 14, \end{aligned}$$

then $f((x_1,x_2)+(y_1,y_2)) \neq f(x_1,x_2) + f(y_1,y_2)$.
(ii)

$$\begin{aligned} f((x_1,x_2)) &= f(cx_1, cx_2) = 3(cx_1) - (cx_2) + 7 \\ &= 3cx_1 + cx_2 + 7 \\ c \cdot f(x_1,x_2) &= c(3x_1 - x_2 + 7) = 3cx_1 + cx_2 + 7c. \end{aligned}$$

Here, $c \cdot f(x_1,x_2) \neq f(c \cdot (x_1,x_2))$.

(8) *Claim*: The function is **linear**.
proof: We verify that f satisfies conditions (i) and (ii):

(i)

$$\begin{aligned} f((x_1,x_2)+(y_1,y_2)) &= f(x_1+y_1,x_2+y_2)\\ &= (3(x_1+y_1)-(x_2+y_2),(x_2+y_2))\\ &= (3x_1+3y_1-x_2-y_2,x_2+y_2)\\ &= (3x_1-x_2,x_2)+(3y_1-y_2,y_2)\\ &= f(x_1,x_2)+f(y_1,y_2), \end{aligned}$$

(ii) $f(c\dot{(}x_1,x_2) = f(cx_1,cx_2) = (3(cx_1)-(cx_2),cx_2) = (c(3x_1-x_2),cx_2) = c\cdot(3x_1-x_2,x_2) = c\cdot f(x_1,x_2)$.

(9) *Claim*: The function is **linear**.
proof: We verify that f satisfies conditions (i) and (ii): (i)

$$\begin{aligned} f(x_1,x_2,x_3)+f(y_1,y_2,y_3) &= f(x_1+y_1,x_2+y_2,x_3+y_3)\\ &= (3(x_1+y_1)-(x_2+y_2)+(x_3+y_3),(x_1+y_1)\\ &\quad -(x_2+y_2)+4(x_3+y_3),4(x_1+y_1)+(x_3+y_3))\\ &= (3x_1+3y_1-x_2-y_2+x_3+y_3,x_1+y_1-x_2-y_2\\ &= \quad +4x_3+4y_3,4x_1+4y_2+x_3+y_3)\\ &= (3x_1-x_2+x_3,x_1-x_2+4x_3,4x_1+x_3)\\ &\quad +(3y_1-y_2+y_3,y_1-y_2+4y_3,4y_1+y_3)\\ &= f(x_1,x_2,x_3)+f(y_1,y_2,y_3), \end{aligned}$$

(ii)

$$\begin{aligned} f(c\cdot x_1,x_2,x_3) &= f(cx_1,cx_2,cx_3)\\ &= (3cx_1-cx_2+cx_3,cx_1-cx_2+4cx_3,4cx_1+cx_3)\\ &= (c(3x_1-x_2+x_3),c(x_1-x_2+4x_3),c(4x_1+x_3))\\ &= c\cdot(3x_1-x_2+x_3,x_1-x_2+4x_3,4x_1+x_3)\\ &= c\cdot f(x_1,x_2,x_3). \end{aligned}$$

$f:\mathbb{R}^3\longrightarrow\mathbb{R}^3,\ f(x_1,x_2,x_3)=(3x_1-x_2+x_3,x_1-x_2+4x_3,4x_1+x_3),$

(10) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii).
(i)

$$\begin{aligned} f(x_1+y_1,x_2+y_2,x_3+y_3) &= (3(x_1+y_1)-(x_2+y_2)+(x_3+y_3),\\ &\quad (x_1+y_1)-(x_2+y_2)+4(x_3+y_3),1)\\ &= (3x_1+3y_1-x_2-y_2+x_3+y_3,x_1+y_1\\ &\quad -x_2-y_2+4x_3+4y_3,1)\\ f(x_1,x_2,x_3)+f(y_1,y_2,y_3) &= (3x_1-x_2+x_3,x_1-x_2+4x_3,1)\\ &\quad +(3x_1-x_2+x_3,x_1-x_2+4x_3,1)\\ &= (3x_1+3y_1-x_2-y_2+x_3+y_3,x_1+y_1\\ &\quad -x_2-y_2+4x_3+4y_3,2). \end{aligned}$$

Here, $f(x_1+y_1,x_2+y_2,x_3+y_3)\neq f(x_1,x_2,x_3)+f(y_1,y_2,y_3)$ because in the third coordinate we have $1\neq 2$.

(i)

$$\begin{aligned} f(c\dot{(}x_1,x_2,x_3)) &= f(cx_1,cx_2,cx_3) \\ &= (3cx_1 - cx_2 + cx_3, cx_1 - cx_2 + 4cx_3, 1) \\ c\cdot f(x_1,x_2,x_3) &= c\cdot(3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, 1) \\ &= (3cx_1 - cx_2 + cx_3, cx_1 - cx_2 + 4cx_3, c) \end{aligned}$$

We have that $f(c\cdot(x_1,x_2,x_3) \neq c\cdot f(x_1,x_2,x_3)$ because in the third coordinate $1 \neq c$, unless $c = 1$.

(11) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii).
(i)

$$\begin{aligned} f(x_1+y_1,x_2+y_2,x_3+y_3) &= (3(x_1+y_1) - (x_2+y_2) + (x_3+y_3), \\ &\quad (x_1+y_1) - (x_2+y_2) + 4(x_3+y_3), \\ &\quad (x_1+y_1)(x_3+y_3), \\ &\quad (x_1+y_1) - (x_2+y_2)) \\ &= (3x_1 + 3y_1 - x_2 - y_2 + x_3 + y_3, \\ &\quad x_1 + y_1 - x_2 + y_2 + 4x_3 + 4y_3, \\ &\quad x_1x_3 + x_1y_3 + y_1x_3 + y_1y_3, \\ &\quad x_1 + y_1 - x_2 - y_2) \\ f(x_1,x_2,x_3) + f(y_1,y_2,y_3) &= (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1x_3, x_1 - x_2) \\ &\quad + (3y_1 - y_2 + y_3, y_1 - y_2 + 4y_3, y_1y_3, y_1 - y_2) \\ &= (3x_1 + 3y_1 - x_2 - y_2 + x_3 + y_3, \\ &\quad x_1 + y_1 - x_2 + y_2 + 4x_3 + 4y_3, \\ &\quad x_1x_3 + y_1y_3, x_1 + y_1 - x_2 - y_2) \end{aligned}$$

Therefore, $f(x_1+y_1,x_2+y_2,x_3+y_3) \neq f(x_1,x_2,x_3) + f(y_1,y_2,y_3)$ due to the difference of the additional term of $x_1y_3 + x_3y_1$ in the third coordinate for $f(x_1+y_1,x_2+y_2,x_3+y_3)$.
(ii)

$$\begin{aligned} f(c\cdot(x_1,x_2,x_3)) &= f(cx_1,cx_2,c_3) \\ &= (3cx_1 - cx_2 + cx_3, cx_1 - cx_2 + 4cx_3, c^2x_1x_3, cx_1 - cx_2) \\ &= (c(3x_1 - x_2 + x_3), c(x_1 - x_2 + 4x_3), c(cx_1x_3), c(x_1 - x_2)) \\ c\cdot f(x_1,x_2,x_3) &= c\cdot(3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1x_3, x_1 - x_2) \\ &= (c(3x_1 - x_2 + x_3), c(x_1 - x_2 + 4x_3), c(x_1x_3), c(x_1 - x_2)) \end{aligned}$$

Here, we see that $f(c\cdot(x_1,x_2,x_3) \neq c\cdot f(x_1,x_2,x_3)$ by the discrepancy in the third coordinate of $c^2 \neq c$ unless $c = \pm 1$.

(12) *Claim*: The function is **non-linear**.
proof: This function fails both conditions (i) and (ii).

(i)

$$\begin{aligned}
f(x_1+y_1, x_2+y_2, x_3+y_3) &= (e^{(x_3+y_3)+(x_1+y_1)}, 3(x_1+y_1)-(x_2+y_2)+(x_3+y_3),\\
&\quad (x_1+y_1)-(x_2+y_2)+4(x_3+y+3), 0)\\
&= (e^{x_3+x_1}e^{y_3+y_1}, 3x_1-x_2+x_3+3y_1-y_2+y_3,\\
&\quad x_1-x_2+4x_3+y_1-y_2+4y_3, 0)\\
f(x_1,x_2,x_3)+f(y_1,y_2,y_3) &= (e^{x_3+x_1}, 3x_1-x_2+x_3, x_1-x_2+4x_3, 0)\\
&+ (e^{y_3+y_1}, 3y_1-y_2+y_3, y_1-y_2+4y_3, 0)\\
&= (e^{x_3+x_1}+e^{y_3+y_1}, 3x_1-x_2+x_3+3y_1-y_2+y_3,\\
&\quad x_1-x_2+4x_3+y_1-y_2+4y_3, 0)\\
f:\mathbb{R}^3 \longrightarrow \mathbb{R}^4,\ f(x_1,x_2,x_3) &= (e^{x_3+x_1}, 3x_1-x_2+x_3, x_1-x_2+4x_3, 0)
\end{aligned}$$

Hence, $f(x_1+y_1, x_2+y_2, x_3+y_3) \neq f(x_1,x_2,x_3)+f(y_1,y_2,y_3)$ for the discrepancy in the first coordinate.

(ii)

$$\begin{aligned}
f(cx_1, cx_2, cx_3) &= (e^{cx_3+cx_1}, 3cx_1-cx_2+cx_3, cx_1-cx_2+4cx_3, 0)\\
c\cdot f(x_1,x_2,x_3) &= (ce^{x_3+x_1}, c(3x_1-x_2+x_3)c, c(x_1-x_2+4x_3), 0)
\end{aligned}$$

Since $e^{c(x_3+x_1)} \neq ce^{x_3+x_1}$ unless $c=1$, then $f(c\cdot(x_1,x_2,x_3) \neq c\cdot f(x_1,x_2,x_3)$. $\square$

Problem 2. For each of the following pairs of maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$, write their composition $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, defined by

$$(g \circ f)(x_1, \dots, x_n) = g(f((x_1, \dots, x_n))).$$

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 3x$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(s) = 4s + 1$.

(2) $f : \mathbb{R} \longrightarrow \mathbb{R}^2$, $f(x) = (2x, 7x)$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g(s,t) = s + 6t$.

(3) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x,y) = (2x + 3y, 7x - y)$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g(s,t) = 3s - t$.

(4) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x,y) = (x - 2y, 4x + 7y, x)$, and the map $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $g(s,t,u) = (s + 3t - u, s + u)$.

Solution.

(1) $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$, where
$(g \circ f)(x) = g(f(x)) = g(3x) = 4(3x) + 1 = 12x + 1.$

(2) $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$, where
$(g \circ f)(x) = g(f(x)) = g((2x, 7x)) = 2x + 6(7x) = 2x + 42x = 44x.$

(3) $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, where

$$\begin{aligned}(g \circ f)(x,y) &= g(f((x,y))) = g((2x + 3y, 7x - y)) = 3(2x + 3y) - (7x - y) \\ &= 6x + 9y - 7x + y = 13x + 10y.\end{aligned}$$

(4) $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, where

$$\begin{aligned}(g \circ f)(x,y) &= g(f((x,y))) = g((x - 2y, 4x + 7y, x)) \\ &= ((x - 2y) + 3(4x + 7y) - x, (x - 2y) + x) \\ &= (x - 2y + 12x + 21y - x, 2x - 2y) = (12x + 19y, 2x - 2y).\end{aligned}$$

□

Problem 3. **Prove**, with an argument, or **disprove**, with a counter-example, each of the statements sentences below.

(1) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are two maps. If f and g are linear, then the composition $g \circ f$ is linear.

(2) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are two maps. If f is linear, then the composition $g \circ f$ is linear.

(3) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are two maps. If f is not linear, then the composition $g \circ f$ is never linear.

(4) For any map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ there exists a linear map $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ such that the composition $g \circ f$ is linear.

(5) For any non-linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ there exists a linear map $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ such that the composition $g \circ f$ is not linear.

Solution.

(1) This statement is **true**.

Proof. Suppose that the maps f, g are linear. First, we want to show that composition of linear maps preserves vector addition, i.e., $(g \circ f)(x + y) = (g \circ f)(x) + (g \circ f)(y)$.

$$\begin{aligned}
(g \circ f)(x + y) &= g(f(x + y)) \\
&= g(f(x) + f(y)) && \text{(by linearity of } f) \\
&= g(f(x)) + g(f(y)) && \text{(by linearity of } g) \\
&= (g \circ f)(x) + (g \circ f)(y).
\end{aligned}$$

Now, we want to show that the composition of linear maps preserves scalar multiplication, i.e., $(g \circ f)(c \cdot x) = c \cdot (g \circ f)(x)$.

$$\begin{aligned}
(g \circ f)(c \cdot x) &= g(f(c \cdot x)) \\
&= g(c \cdot f(x)) && \text{(by linearity of } f) \\
&= c \cdot g(f(x)) && \text{(by linearity of } g) \\
&= c \cdot (g \circ f)(x).
\end{aligned}$$

Therefore, since $(g \circ f)(x)$ satisfies the conditions of scalar multiplication and vector addition then the map is linear. □

(2) This statement is **false**.

Counterexample: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ where $f(x_1, x_2) = x_1 + x_2$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ where $g(x) = e^x$. Then the composition map $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined $(g \circ f)(x_1, x_2) = e^{x_1 + x_2}$ violates both scalar multiplication and vector addition since

$$(g \circ f)(x_1 + y_1, x_2 + y_2) = e^{x_1 + x_2 + y_1 + y_2} \neq e^{x_1 + x_2} + e^{y_1 + y_2} = (g \circ f)(x) + (g \circ f)(y),$$
$$(g \circ f)(c \cdot x) = e^{c(x_1 + x_2)} \neq ce^{x_1 + x_2} = c \cdot (g \circ f)(x).$$

(3) This statement is **false**.

Counterexample: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = e^x$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ where $g(x) = \ln x$. Then the composition map $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $(g \circ f)(x) = x$, which is clearly linear. To check this

$$(g \circ f)(x + y) = x + y = (g \circ f)(x) + (g \circ f)(y),$$
$$(g \circ f)(cx) = cx = c \cdot (g \circ f)(x).$$

(4) This statement is **true**.

Proof. Let $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ be the zero map. Then the composition map $(g \circ f)(x) = 0 \in \mathbb{R}^k$. We have that the zero map is trivially linear because

$$(g \circ f)(x + y) = g(f(x + y)) = 0 = 0 + 0 = (g \circ f)(x) + (g \circ f)(y)$$
$$(g \circ f)(c \cdot x) = 0 = c \cdot 0 = c \cdot (g \circ f)(x)$$

□

Note that if the problem statement had asked for a nontrivial map g, then this statement would be false. In this case, the function f could be some combination of the linear and non-linear terms, making it impossible for the function g to resolve the non-linear terms without creating new non-linear terms out of the linear terms from f.

(5) The statement is **true**.

Proof. If we suppose $k = m$, then let g be the identity map. Therefore, $g \circ f = f$ which is non-linear by definition.

Otherwise, let $f(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$ and let $y_i = f_i(x_1, x_2, \ldots, x_n)$ for $1 \leq i \leq m$ be the function which is non-linear. There may be more than one non-linear function, here we choose one.

Now, suppose $k \neq m$, i.e. $k < m$ or $k > m$, then let $g(y_1, \ldots, y_m) = (y_i, 0, \ldots, 0)$, in other words, let g be the identity map on the coordinate associated to the non-linear equation and 0 for the remaining $|k - m|$ coordinates, we call this function the *projection* map. Here, the map g is linear since it satisfies vector addition and scalar multiplication:

$$\begin{aligned}
g(x_1 + x_1', \ldots, x_m + x_m') &= (x_i + x_i', 0 \ldots, 0) \\
&= (x_i, 0, \ldots, 0) + (x_i', 0, \ldots, 0) \\
&= g(x_1, \ldots, x_m) + g(x_1', \ldots, x_m') \\
g(c \cdot (x_1, \ldots, x_m)) &= g(cx_1, cx_2, \ldots, cx_n) \\
&= (cx_i, 0, \ldots, 0)) \\
&= c((x_i, 0, \ldots, 0) \\
&= c \cdot g(x_1, x_2, \ldots, x_n).zse3
\end{aligned}$$

However, the composition map which is defined

$$(g \circ f)(x_1, \ldots, x_n) = (f_i(x_1, \ldots, x_n), 0, \ldots, 0)$$

is non-linear. □

□

Problem 4. Suppose that a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$.

(1) Show that $f(n \cdot x) = n \cdot f(x)$ for all natural numbers $n \in \mathbb{N}$.
(2) Show that $f(q \cdot x) = q \cdot f(x)$ for all rational numbers $q \in \mathbb{Q}$.

In particular, a continuous function satisfying condition (i) of linearity also satisfies condition (ii).

Solution.

(1) We show that $f(n \cdot x) = n \cdot f(x)$ for all natural numbers n using a recursive argument.
First, we see that $f(1 \cdot x) = f(x) = 1 \cdot f(x)$ and for $n = 2$, $f(2 \cdot x) = f(x + x) = f(x) + f(x) = 2f(x)$. Similarly for $n = 3$, $f(3 \cdot x) = f(2x + x) = f(2x) + f(x) = 2f(x) + f(x) = 3f(x)$.
Since the natural numbers are defined recursively, i.e., if $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$, let's now suppose that the statement holds for some arbitrary n, i.e., $f(n \cdot x) = n \cdot f(x)$, and we'll show that $f((n+1) \cdot x) = (n+1) \cdot f(x)$.

$$f((n+1) \cdot x) = f(n \cdot x + x) = n \cdot f(x) + f(x) = (n+1) \cdot f(x).$$

Now, we have shown that for any natural number $n \in \mathbb{N}$ that $f(n \cdot x) = n \cdot f(x)$.
This type of argument is called an inductive proof which works for showing that a statement holds for a natural number and can be generalized to the integers. The general procedure to show an inductive proof is you show that the statement holds for a "base case" typically 1 but can be for any integer k. Then you perform the "inductive hypothesis" step which is when you assume that the statement holds for some particular natural number n and then you show that the statement holds for $n + 1$.

(2) Let $q = \frac{p}{r} \in \mathbb{Q}$ where $p \in \mathbb{Z}$ and $r \in \mathbb{N}$. First we want to show that $f(\frac{1}{r} \cdot x) = \frac{1}{r} f(x)$, using the results from part (1) we see that

$$f(x) = f(r \cdot \frac{1}{r} x) = r \cdot f(\frac{1}{r} x)$$

Therefore, $f(x) = r \cdot f(\frac{1}{r} x)$ and since $r \neq 0$, then we may divide by r to obtain $\frac{1}{r} f(x) = f(\frac{1}{r} x)$.
Now, to show the desired statement, we initially use the results from part (1) then the result from above,

$$f(q \cdot x) = f(\frac{p}{r} \cdot x) = f(p \frac{1}{r} x) = p f(\frac{1}{r} x) = p \frac{1}{r} f(x) = q f(x).$$

□