LECTURE 6: PRACTICE EXERCISES

MAT-67 SPRING 2024

Abstract. These practice problems correspond to the 6th lecture of MAT-67 Spring 2024, delivered on April 12th 2024.

The following are practice problems. They are not to be submitted, they are for your own practice. Solutions will be posted soon.

Problem 1. Consider the following subspace $U_1, U_2 \subseteq V$. In each example, compute their intersection $U_1 \cap U_2$, their sum and decide whether the sum is a direct sum or not.

(1) The subspaces $U_1, U_2 \subseteq V = \mathbb{R}^2$ given by

$$U_1 = \{(x_1, x_2) \in V : x_1 + 4x_2 = 0\} \subseteq V,$$
$$U_2 = \{(x_1, x_2) \in V : 3x_1 - x_2 = 0\} \subseteq V.$$

(2) The subspaces $U_1, U_2 \subseteq V = \mathbb{R}^2$ where U_1 is the unique subspace containing the vector $(1,3) \in \mathbb{R}^2$ (and not equal to V) and U_2 is the unique subspace containing the vector $(-2,7) \in \mathbb{R}^2$ (and not equal to V).

(3) The subspaces $U_1, U_2 \subseteq V = \mathbb{R}^2$ where U_1 is the unique subspace containing the vector $(1,3) \in \mathbb{R}^2$ (and not equal to V) and U_2 is the unique subspace containing the vector $(-2,-6) \in \mathbb{R}^2$ (and not equal to V).

(4) The subspaces $U_1, U_2 \subseteq V = \mathbb{R}^3$ where U_1 is the unique subspace containing the vectors $(1,3,-2), (0,5,7) \in \mathbb{R}^2$ (and not equal to V) and $U_2 = \{x_1 = 0, x_2 + x_3 = 0\}$.

(5) The subspaces $U_1, U_2 \subseteq V = \mathbb{R}^4$ given by

$$U_1 = \{(x_1, x_2, x_3, x_4) \in V : x_1 + 4x_2 - x_3 = 0\} \subseteq V,$$
$$U_2 = \{(x_1, x_2, x_3, x_4) \in V : 3x_1 - x_2 = 0, x_4 = 0\} \subseteq V.$$

(6) The subspaces $U_1, U_2 \subseteq V = \mathbb{R}^4$ given by

$$U_1 = \{(x_1, x_2, x_3, x_4) \in V : x_1 + 4x_2 - x_3 = 0, x_4 = 0\} \subseteq V,$$
$$U_2 = \{(x_1, x_2, x_3, x_4) \in V : 3x_1 - x_2 = 0, x_2 + x_4 = 0, x_1 + x_3 = 0\} \subseteq V.$$

(7) The subspaces $U_1, U_2 \subseteq V = \mathbb{R}[x]$ given by

$$U_1 = \{p(x) \in \mathbb{R}[x] : p(0) = 0\},$$
$$U_2 = \{p(x) \in \mathbb{R}[x] : p(3) = 0\}$$

Problem 2. Consider the two subspaces $U_1, U_2 \subseteq V = \mathbb{R}^3$ given by

$$U_1 = \{(x_1, x_2, x_3) \in V : x_1 + 3x_2 = 0\}$$
$$U_2 = \{(x_1, x_2, x_3) \in V : x_1 + x_2 + x_3 = 0\}.$$

Show that $U_1 \cap U_2 = W$, where W is the unique vector subspace $W \subseteq V$ that contains the vector $(3, -1, -2) \in V$ but W is not equal to V.

Problem 3. Let $U \subseteq V$ be the subspace of $V = \mathbb{R}^4$ given by

$$U = \{(x_1, x_2, x_3, x_4) \in V : x_1 + 3x_2 - x_4 = 0, 5x_3 + x_4 = 0\}.$$

Find two *different* subspaces $W_1, W_2 \subseteq V$ such that $U \oplus W_1 = V$ and $U \oplus W_2 = V$.

Problem 4. Let $U \subseteq V$ be the subspace of $V = \mathbb{R}^5$ given by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in V : x_1 + 3x_2 - x_4 + x_5 = 0\}.$$

Show that for any vector $v \in V$ such that $v \notin U$, then $V = U \oplus W_v$ where W_v is the subspace $W_v = \{w \in V : w = \alpha \dot{v}, \text{for some } \alpha \in \mathbb{R}\}$.