LECTURE 7: PRACTICE EXERCISES

MAT-67 SPRING 2024

Abstract. These practice problems correspond to the 7th lecture of MAT-67 Spring 2024, delivered on April 15th 2024.

The following are practice problems. They are not to be submitted, they are for your own practice. Solutions will be posted soon.

Recall: Let V be an $\mathbb{R}$-vector space. Given a vector $w \in V$ and a set of vectors $\{v_1, \ldots, v_k\} \in V$, we say that w is a linear combination of $\{v_1, \ldots, v_k\}$ if there exists real constants $a_1, \ldots, a_k \in \mathbb{R}$ such that

$$w = a_1 v_1 + \ldots + a_k v_k.$$

Also, independently, recall that the space of $\{v_1, \ldots, v_k\}$ is the subspace $span(v_1, \ldots, v_k) \subseteq V$ that contains *all* linear combinations of $\{v_1, \ldots, v_k\}$. We proved in lecture that this is the smallest subspaces containing each v_i, $1 \leq i \leq k$.

Problem 1. For each of the following vectors $w \in V$ and as set of vectors $\{v_1, \ldots, v_k\} \in V$, decide whether w is a linear combination of $\{v_1, \ldots, v_k\}$.

(1) Let $V = \mathbb{R}^2$, $w = (1, 2)$ and
$$\{v_1, v_2\} = \{(1, 0), (1, 1)\}.$$
(2) Let $V = \mathbb{R}^2$, $w = (1, 2)$ and
$$\{v_1, v_2\} = \{(-2, -4), (1, 1)\}.$$
(3) Let $V = \mathbb{R}^2$, $w = (1, 2)$ and
$$\{v_1, v_2\} = \{(3, -5), (12, -20)\}.$$
(4) Let $V = \mathbb{R}^3$, $w = (-3, 1, 4)$ and
$$\{v_1, v_2\} = \{(1, 0, 3), (1, 1, 4)\}.$$
(5) Let $V = \mathbb{R}^3$, $w = (-3, 2, 4)$ and
$$\{v_1, v_2\} = \{(-3, 0, -2), (0, 1, 3)\}.$$
(6) Let $V = \mathbb{R}^3$, $w = (-3, 2, 4)$ and
$$\{v_1, v_2, v_3\} = \{(-1, 0, -1), (0, 1, 2), (0, 0, 1)\}.$$
(7) Let $V = \mathbb{R}[x]$, $w = 3 - x$ and
$$\{v_1, v_2\} = \{1, 2 - x^2\}.$$
(8) Let $V = \mathbb{R}[x]$, $w = 3 - x + 7x^3 - x^6$ and
$$\{v_1, v_2, v_3\} = \{1, 2 - x, x^3, x^5 - x^3 - 8\}.$$
(9) Let $V = \mathbb{R}[x]$, $w = 3 - x + 7x^3 - x^6$ and
$$\{v_1, v_2, v_3\} = \{1, 2 - x, x^3, x^5 - x^3 - 8, x^6 + 4x\}.$$

Problem 2. Given a subset $\{v_1, \dots, v_k\}$, $v_i \in V$ for $1 \leq i \leq k$, and a vector subspace $U \subseteq V$, prove or disprove whether $span(v_1, \dots, v_k) = U$.

(1) $V = \mathbb{R}^2$ and
$$\{v_1\} = \{(1, -3)\}$$
and the vector subspace
$$U = \{3x_1 - x_2 = 0\}.$$

(2) $V = \mathbb{R}^2$ and
$$\{v_1\} = \{(1, -3)\}$$
and the vector subspace
$$U = \{3x_1 + x_2 = 0\}.$$

(3) $V = \mathbb{R}^3$ and
$$\{v_1, v_2\} = \{(1, -3, 2), (4, 5, 0)\}$$
and the vector subspace
$$U = \{5x_1 + x_2 - x_3 = 0\}.$$

(4) $V = \mathbb{R}^3$ and
$$\{v_1, v_2\} = \{(1, -3, 2), (0, 1, 1)\}$$
and the vector subspace
$$U = \{5x_1 + x_2 - x_3 = 0\}.$$

(5) $V = \mathbb{R}^4$ and
$$\{v_1, v_2\} = \{(1, -3, 2, 0), (2, 0, 1, 1)\}$$
and the vector subspace
$$U = \{8x_1 + 5x_2 - x_3 + 17x_4 = 0\}.$$

(6) $V = \mathbb{R}^4$ and
$$\{v_1, v_2\} = \{(1, 0, 2, 0), (2, 0, 1, 1)\}$$
and the vector subspace
$$U = \{x_2 = 0, -2x_1 + x_3 + 3x_4 = 0\}.$$

(7) $V = \mathbb{R}^4$ and
$$\{v_1, v_2, v_3\} = \{(1, -3, 2, 0), (2, 0, 1, 1), (0, 0, 0, 1)\}$$
and the vector subspace
$$U = \{8x_1 + 5x_2 - x_3 + 17x_4 = 0\}.$$

Problem 3. Given two subsets $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_s\}$, $v_i, w_j \in V$, $1 \le i \le k$ and $1 \le j \le s$, prove or disprove whether $span(v_1, \dots, v_k) = span(w_1, \dots, w_k)$.

(1) Let $V = \mathbb{R}^2$ and
$$\{v_1\} = \{(2, -3)\},$$
$$\{w_1\} = \{(1, -3)\}.$$

(2) Let $V = \mathbb{R}^3$ and
$$\{v_1\} = \{(2, -4, 6)\},$$
$$\{w_1\} = \{(1, -2, 3)\}.$$

(3) Let $V = \mathbb{R}^3$ and
$$\{v_1, v_2\} = \{(2, -4, 6), (1, 0, 0)\},$$
$$\{w_1, w_2\} = \{(1, -2, 3), (0, 0, 1)\}.$$

(4) Let $V = \mathbb{R}^3$ and
$$\{v_1, v_2\} = \{(1, 2, -3), (-5, 6, -1)\},$$
$$\{w_1, w_2\} = \{(4, 0, -4), (1, -2, 1)\}.$$

Problem 4. Solve the following parts:

(1) Find an example of 3 vectors $v_1, v_2, v_3 \in \mathbb{R}^4$ and of 3 vectors $w_1, w_2, w_3 \in \mathbb{R}^4$ such that $span(v_1, v_2, v_3) \ne span(w_1, w_2, w_3)$ and $span(v_1, v_2) = span(w_1, w_2)$.

(2) Do there exist 3 vectors $v_1, v_2, v_3 \in \mathbb{R}^4$ and 3 vectors $w_1, w_2, w_3 \in \mathbb{R}^4$ such that $span(v_1, v_2, v_3) \ne span(w_1, w_2, w_3)$ but
$$span(v_1, v_2) = span(w_1, w_2), \quad span(v_1, v_3) = span(w_1, w_3),$$
$$\text{and } span(v_3, v_2) = span(w_3, w_2)?$$

(3) Suppose that $w \in V$ is not a linear combination of $\{v_1, \dots, v_k\}$. Show that $a \cdot w \in V$ satisfies $a \cdot w \notin span(v_1, \dots, v_k)$ for all $a \in \mathbb{R}$ non-zero.