LECTURE 8: PRACTICE EXERCISES

MAT-67 SPRING 2024

ABSTRACT. These practice problems correspond to the 8th lecture of MAT-67 Spring 2024, delivered on April 17th 2024.

The following are practice problems. They are not to be submitted, they are for your own practice. Solutions will be posted soon.

Recall: Let V be an $\mathbb{R}$-vector space. Given a vector $w \in V$ and a set of vectors $\{v_1, \ldots, v_k\} \in V$, we say that w is a linear combination of $\{v_1, \ldots, v_k\}$ if there exists real constants $a_1, \ldots, a_k \in \mathbb{R}$ such that

$$w = a_1 v_1 + \ldots + a_k v_k.$$

Also, independently, recall that the space of $\{v_1, \ldots, v_k\}$ is the subspace $span(v_1, \ldots, v_k) \subseteq V$ that contains *all* linear combinations of $\{v_1, \ldots, v_k\}$. We proved in lecture that this is the smallest subspaces containing each v_i, $1 \leq i \leq k$.

Problem 1. Solve the following parts:

(1) Let $V = \mathbb{R}^4$. Give an example of a vector $w \in \mathbb{R}^4$ and a subset $\{v_1, v_2, v_3\}$, $v_i \in V$, such that w is a linear combination of v_1, v_2, v_3 in a unique way.

(2) Let $V = \mathbb{R}^4$. Does there exist a vector $w \in \mathbb{R}^4$ and a subset $\{v_1, v_2, v_3\}$, $v_i \in V$, such that w is a linear combination of v_1, v_2, v_3 in at least *two* different ways?

(3) Let $V = \mathbb{R}^4$. Does there exist a vector $w \in \mathbb{R}^4$ and a subset $\{v_1, v_2, v_3\}$, $v_i \in V$ such that w is a linear combination of v_1, v_2, v_3 in at *infinitely many* different ways?

Problem 2. For each of the following subspaces $U \subseteq V$, describe a set of vector $\{u_1, \ldots, u_k, \ldots\} \in V$ such that $U = span(u_1, \ldots, u_k, \ldots)$. (Note that the set might have to be infinite.)

(1) Let $V = \mathbb{R}^2$ and $U = \{x_1 - 3x_2 = 0\}$.

(2) Let $V = \mathbb{R}^3$ and $U = \{x_1 - 4x_2 + 2x_3 = 0\}$.

(3) Let $V = \mathbb{R}^3$ and $U = \{x_1 - 4x_2 + 2x_3, x_2 - 6x_3 = 0\}$.

(4) Let $V = \mathbb{R}^4$ and $U = \{x_1 - x_2 - 2x_3 + 6x_4 = 0\}$.

(5) Let $V = \mathbb{R}^4$ and $U = \{x_1 - x_2 - 2x_3 + 6x_4 = 0, x_1 + x_2 + x_3 = 0\}$.

(6) Let $V = \mathbb{R}^4$ and $U = \{x_1 - x_2 - 2x_3 + 6x_4 = 0, x_1 + x_2 + x_3 = 0, x_1 - 7x_3 - x_4 = 0\}$.

(7) Let $V = \mathbb{R}[x]$ and $U = \{p(x) \in \mathbb{R}[x] : p(0) = 0\}$.

(8) Let $V = \mathbb{R}[x]$ and $U = \{p(x) \in \mathbb{R}[x] : p(0) = 0, p(1) = 0\}$.

(9) Let $V = \mathbb{R}[x]$ and $U = \{p(x) \in \mathbb{R}[x] : p(0) = 0, p(1) = 0, p(-1) = 0\}$.

Problem 3. Let $V = \mathbb{R}^n$. Suppose that $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_{n-k}\}$, $v_i, w_j \in V$, are such that

$$v_i \notin span(w_1, \ldots, w_{n-k}), \quad \forall 1 \leq i \leq k, \text{ and}$$
$$w_j \notin span(v_1, \ldots, v_k), \quad \forall 1 \leq j \leq n-k.$$

(1) Show that the following sum is a direct sum:

$$span(v_1, \ldots, v_k) + span(w_1, \ldots, w_{n-k}) = span(v_1, \ldots, v_k) \oplus span(w_1, \ldots, w_{n-k}).$$

(2) Suppose that all the v_i are linearly independent among them, and all the w_j are linearly independent among them. Show that

$$V = span(v_1, \ldots, v_k) \oplus span(w_1, \ldots, w_{n-k}).$$

Problem 4. Let $V = \mathbb{R}^4$. Solve the following parts.

(1) Give an example of two subsets $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$, $v_i, w_j \in V$ such that

$$V = span(v_1, v_2, v_3) + span(w_1, w_2)$$

but $V \neq span(v_1, v_2, v_3) \oplus span(w_1, w_2)$.

(2) Give an example of two subsets $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$, $v_i, w_j \in V$ such that

$$V = span(v_1, v_2, v_3) \oplus span(w_1, w_2).$$

(3) Give an example of two subsets $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$, $v_i, w_j \in V$ such that

$$V \neq= span(v_1, v_2, v_3) + span(w_1, w_2).$$