University of California Davis **Name (Print):** ______________________

Abstract Linear Algebra MAT 67 **Student ID (Print):** ______________________

Practice Midterm Examination **May 31st 2024**

Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^3$ and consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A_f = \begin{pmatrix} 5 & -3 & -4 \\ -3 & -1 & 5 \\ -5 & 3 & 4 \end{pmatrix}$$

in the standard coordinate basis.

(a) (5 points) Find the eigenvalues of A_f.

The characteristic polynomial of A_f is

$$\det(A_f) = \det \begin{pmatrix} 5-\lambda & -3 & -4 \\ -3 & -1-\lambda & 5 \\ -5 & 3 & 4-\lambda \end{pmatrix} = -\lambda^3 + 8\lambda^2 + 33\lambda = -\lambda(\lambda+3)(\lambda-11).$$

Therefore the eigenvalues are $\lambda = 0, -3$ and 11.

(b) (15 points) Find a basis of eigenvectors for A_f.

Since the eigenvalues are different, any three eigenvectors will be a basis.

For a given λ, we must find an eigenvector v_λ, which is a non-zero vector in the kernel $\ker(A_f - \lambda \cdot \mathrm{Id})$. For instance, we can pick

$$v_0 = (19, 13, 14), \quad v_{-3} = (-1, -4, 1), \quad v_{11} = (-3, 2, 3).$$

(c) (5 points) Write $A = S \cdot D \cdot S^{-1}$ where D is a diagonal matrix.

By Part (a), the matrix D can be written as

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 11 \end{pmatrix}.$$

The matrix S is to be chosen as the matrix of eigenvectors, written in columns and in the same order as the eigenvalues. Therefore we get

$$S = \begin{pmatrix} 19 & -1 & -3 \\ 13 & -4 & 2 \\ 14 & 1 & 3 \end{pmatrix}.$$

By construction, $A = S \cdot D \cdot S^{-1}$.

2. (25 points) Consider the vector space $V = \mathbb{R}^3$ and consider the $\mathbb{R}$-bilinear binary operation $\langle \cdot, \cdot \rangle_A : V \times V \to \mathbb{R}$ given by

$$\langle v, w \rangle_f = v^t \cdot A \cdot w, \quad \text{where } A = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 0 \\ 4 & 0 & 5 \end{pmatrix}.$$

(a) (10 points) Show that $\langle \cdot, \cdot \rangle_A$ is an inner product.

First, A is symmetric. Thus it suffices to argue that A is positive definite. This can be done by finding the eigenvalues of A and checking that they are positive. The eigenvalues are $1, 5$ and 9, so that is true. Alternatively, Sylvester's criterion, as stated in class, shows that A is positive definite if and only if all the principal minors are positive. The principal minors are 5, $25 = 5 \cdot 5$ and $\det(A) = 45$, so all positive.

(b) (5 points) Compute the length of the vector $v = (1, 1, 1)$.

By definition, the square of the length is

$$\langle v, v \rangle_A = v^t A v = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 0 \\ 4 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 5 \\ 9 \end{pmatrix} = 23.$$

So the length is $\sqrt{23}$.

(c) (5 points) Show that $(1, 0, 0)$ and $(0, 0, 1)$ are *not* orthogonal with respect to $\langle \cdot, \cdot \rangle_A$.

By definition, orthogonal means $\langle (1, 0, 0), (0, 0, 1) \rangle_A = 0$, and we have

$$\langle (1,0,0),(0,0,1)\rangle_A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 0 \\ 4 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 4.$$

(d) (4 points) Find a non-zero vector w which is orthogonal to $(1, 0, 0)$.

We can take, for instance, $(0, 1, 0)$. Indeed, we have:

$$\langle (1,0,0),(0,1,0)\rangle_A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 0 \\ 4 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.$$

3. (25 points) Let V be a finite-dimensional $\mathbb{R}$-vector space and write $n = \dim(V)$.

   (a) (15 points) Show that $f : V \to V$ is invertible iff 0 is not an eigenvalue of f.

   By lecture, f is invertible if and only if its determinant is non-zero. Since the determinant is the product of eigenvalues, this implies that f is invertible iff the product of eigenvalues is non-zero. Therefore, this holds iff each eigenvalue is itself non-zero.

   (b) (10 points) Show that $\det(c \cdot f) = c^n \det(f)$, if $c \in \mathbb{R}$ and $f : V \to V$ linear.

   The determinant measure the n-dimensional volume of the image of the standard n-dimensional cube. Since n-dimensional volume scales by a constant $c \in \mathbb{R}_+$ according to the nth power c^n, the statement follows.

   Alternatively, the eigenvalues of $c \cdot f$ are c times the eigenvalues $\{\lambda_i\}$ of f, i.e. they are $\{c \cdot \lambda_i\}$. Since the determinant is the product of the eigenvalues and there are n eigenvalues, the determinant of $c \cdot f$ is c^n times the determinant of f.

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) A square matrix with all eigenvalues positive defines an inner product.

   (1) True. (2) **False**.

   (b) (5 points) If A is an invertible matrix, the eigenvalues of its inverse matrix A^{-1} must be minus the eigenvalues of A.

   (1) True. (2) **False**.

   (c) (5 points) If $f : V \to W$ is a linear map, $\dim(W) = \dim(\ker(f)) + \dim(\operatorname{im}(f))$.

   (1) True. (2) **False**.

   (d) (5 points) Let $f : V \to W$ and $g : W \to U$ be linear maps, then $\ker(f) \subseteq \ker(g \circ f)$.

   (1) **True**. (2) False.

   (e) (5 points) The matrix $A = \begin{pmatrix} 2 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$ is invertible.

   Hint: there are simpler ways than computing the determinant by brute force.

   (1) True. (2) **False**.