**University of California Davis** **Name (Print):** ____________________

**Abstract Linear Algebra MAT 67** **Student ID (Print):** ____________________

**Practice Midterm Examination** **May 31st 2024**

**Time Limit: 50 Minutes**

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This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| 3 | 25 | |
| 4 | 25 | |
| Total: | 100 | |

1. (25 points) Let $V = \mathbb{R}^4$ and consider the linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$ given by the matrix

$$A_f = \begin{pmatrix} 0 & 1 & 2 & -1 \\ -1 & 5 & 4 & 0 \\ 2 & 4 & 5 & 1 \\ 0 & 2 & 11 & -9 \end{pmatrix}$$

in the standard coordinate basis.

(a) (5 points) Show that $\dim(\ker(f)) = 1$.

(b) (10 points) Find a basis of the kernel $\ker(f)$.

(c) (5 points) Find a basis of the image $\mathrm{im}(f)$.

(d) (5 points) Prove that $f$ is *not* invertible.

2. (25 points) Consider the vector space $V = \mathbb{R}^3$ and consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A_f = \begin{pmatrix} 4 & 0 & -3 \\ 1 & 4 & -2 \\ -1 & -2 & 4 \end{pmatrix}$$

in the standard coordinate basis.

(a) (8 points) Show that $v_7 = (-1, -1, 1)$ is an eigenvector with eigenvalue $\lambda = 7$.

(b) (5 points) Prove that $\lambda = 2$ and $\lambda = 3$ are eigenvalues of $f$.

(c) (8 points) Find an eigenvector $v_2$ with eigenvalue $\lambda = 2$ and an eigenvector $v_3$ with eigenvalue $\lambda = 3$.

(d) (4 points) Show that $f$ is injective.

3. (25 points) Consider $V = \mathbb{R}^2$ and the linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$A_f = \begin{pmatrix} 2 & 0 & 8 \\ -1 & 7 & 6 \end{pmatrix}$$

in the standard coordinate basis. Consider the vectors $v_1 = (1, 0, 1)$, $v_2 = (0, 2, 0)$ and $v_3 = (-1, 0, 3)$ in $\mathbb{R}^3$ and the vectors $w_1 = (1, -2)$ and $w_2 = (0, 1)$ in $\mathbb{R}^2$.

(a) (10 points) Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ and $\{w_1, w_2\}$ is a basis of $\mathbb{R}^2$.

(b) (15 points) Find the matrix of $f$ in the bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$.

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) If a $(2 \times 2)$ matrix has all eigenvalues equal to 1, then it must be equal to the $(2 \times 2)$ identity matrix.

   (1) True. (2) False.

   (b) (5 points) If $A = SDS^{-1}$, then $A^3 = S^3D^3S^{-3}$.

   (1) True. (2) False.

   (c) (5 points) The product of two diagonal matrices is again diagonal.

   (1) True. (2) False.

   (d) (5 points) For any $(2 \times 2)$ matrices $A, B$, $e^{A+B} = e^A \cdot e^B$.

   (1) True. (2) False.

   (e) (5 points) Consider a linear function $f : \mathbb{R}^{24} \to \mathbb{R}^{24}$ such that $\dim(\mathrm{im}(f)) = 21$. Then $f$ cannot be injective.

   (1) True. (2) False.