University of California Davis **Name (Print):** ____________________
Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Practice Midterm Examination **May 31st 2024**
Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^4$ and consider the linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$ given by the matrix

$$A_f = \begin{pmatrix} 0 & 1 & 2 & -1 \\ -1 & 5 & 4 & 0 \\ 2 & 4 & 5 & 1 \\ 0 & 2 & 11 & -9 \end{pmatrix}$$

in the standard coordinate basis.

(a) (5 points) Show that $\dim(\ker(f)) = 1$.

By the dimension formula, $\dim(\ker(f)) = 4 - \dim(\text{im}(f))$. The dimension $\dim(\text{im}(f))$ is given by the size of the largest square submatrix with non-zero determinant. Note that $\det(A_f) = 0$, either by direct computation or noting that that adding 1st column, 2nd column and subtractic 3rd column yields the 4th column (so there is linear dependence of columns).

Now, since $\det(A_f) = 0$, $\dim(\text{im}(f)) \leq 3$. For a lower bound, note that the determinant of the upper left 3×3 principal minor is non-zero, so $3 \leq \dim(\text{im}(f))$. Thus $\dim(\text{im}(f)) = 3$ and $\dim(\ker(f)) = 1$ by the dimension formula.

(b) (10 points) Find a basis of the kernel $\ker(f)$.

By Part (a), any non-zero vector in the kernel $\ker(f)$ is a basis, as $\dim(\ker(f)) = 1$. For instance, we can take $v \in \ker(f)$ to be $v = (1, 1, -1, -1)$. A basis for $\ker(f)$ is thus $\{v\}$.

(c) (5 points) Find a basis of the image $\mathrm{im}(f)$.

Since the determinant of the upper left 3×3 principal minor is non-zero, the first three columns of A_f span $\mathrm{im}(f)$. By Part (a), $\dim(\mathrm{im}(f)) = 3$, so they must be a basis. So a basis of $\mathrm{im}(f)$ is $\{v_1, v_2, v_3\}$ where $v_1 = (0, -1, 2, 0)$, $v_2 = (1, 5, 4, 2)$, $v_3 = (2, 4, 5, 11)$.

(d) (5 points) Prove that f is *not* invertible.

A linear map is invertible iff $\det(f) \neq 0$. As explained in the solution of Part (a), $\det(f) = 0$. Alternatively, since $\ker(f)$ is non-trivial, 0 is an eigenvalue of A_f. The determinant of f is the product of eigenvalues, so the determinant is zero and thus f is not invertible.

2. (25 points) Consider the vector space $V = \mathbb{R}^3$ and consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A_f = \begin{pmatrix} 4 & 0 & -3 \\ 1 & 4 & -2 \\ -1 & -2 & 4 \end{pmatrix}$$

in the standard coordinate basis.

(a) (8 points) Show that $v_7 = (-1, -1, 1)$ is an eigenvector with eigenvalue $\lambda = 7$.

It suffices to check that $\det(A_f - 7\mathrm{Id}) = 0$. This is

$$\det \begin{pmatrix} 4-7 & 0 & -3 \\ 1 & 4-7 & -2 \\ -1 & -2 & 4-7 \end{pmatrix} = \det \begin{pmatrix} -3 & 0 & -3 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{pmatrix} = 0$$

which is 0 because the sum of the first two columns gives you the third column. (Alternatively, you can also do direct computation of the determinant.)

(b) (5 points) Prove that $\lambda = 2$ and $\lambda = 3$ are eigenvalues of f.

As in Part (a), we can check that $\det(A_f - 2\mathrm{Id}) = 0$ and $\det(A_f - 3\mathrm{Id}) = 0$. Alternatively, it suffices to check that 2 and 3 are roots of the characteristic polynomial. The characteristic polynomial of A_f is $\det(A_f - \lambda \cdot \mathrm{Id})$, which is

$$\det \begin{pmatrix} 4-\lambda & 0 & -3 \\ 1 & 4-\lambda & -2 \\ -1 & -2 & 4-\lambda \end{pmatrix} = -\lambda^3 + 12\lambda^2 - 41\lambda + 42 = -(\lambda - 7)(\lambda - 2)(\lambda - 3),$$

and thus $\lambda = 2, 3$ are roots of the characteristic polynomial and so eigenvalues of A_f.

(c) (8 points) Find an eigenvector v_2 with eigenvalue $\lambda = 2$ and an eigenvector v_3 with eigenvalue $\lambda = 3$.

For v_2, we want a non-zero vector in the kernel of

$$\begin{pmatrix} 4-2 & 0 & -3 \\ 1 & 4-2 & -2 \\ -1 & -2 & 4-2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}.$$

For instance, we can take $v_2 = (6, 1, 4)$.

For v_3, we want a non-zero vector in the kernel of

$$\begin{pmatrix} 1 & 0 & -3 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}.$$

For instance, we can take $v_3 = (3, -1, 1)$.

(d) (4 points) Show that f is injective.

A linear map $f : V \to V$ is injective iff $\det(f) \neq 0$. Since the eigenvalues of f are $2, 3$ and 7, its determinant is $2 \cdot 3 \cdot 7 \neq 0$, so f is injective.

3. (25 points) Consider $V = \mathbb{R}^2$ and the linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$A_f = \begin{pmatrix} 2 & 0 & 8 \\ -1 & 7 & 6 \end{pmatrix}$$

in the standard coordinate basis. Consider the vectors $v_1 = (1, 0, 1)$, $v_2 = (0, 2, 0)$ and $v_3 = (-1, 0, 3)$ in $\mathbb{R}^3$ and the vectors $w_1 = (1, -2)$ and $w_2 = (0, 1)$ in $\mathbb{R}^2$.

(a) (10 points) Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ and $\{w_1, w_2\}$ is a basis of $\mathbb{R}^2$.

The vectors v_1, v_2, v_3 are linearly independent iff the determinant

$$\det \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

is non-zero. This determinant, by direct computation (e.g. develop the second row), is non-zero (it should be equal to 4). Since v_1, v_2, v_3 are 3 linearly independent vectors in $\mathbb{R}^3$, they are a basis.

Similarly, the determinant

$$\det \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = 1$$

is non-zero, so w_1, w_2 are linearly independent and thus a basis of $\mathbb{R}^2$.

(b) (15 points) Find the matrix of f in the bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$.

The ith column of this matrix is the vector $f(v_i)$ expressed in the basis $\{w_1, w_2\}$. We have

$$f(v_1) = (10, 5), \quad f(v_2) = (0, 14), \quad f(v_3) = (22, 17)$$

in the coordinate $\{(1, 0), (0, 1)\}$ basis of $\mathbb{R}^2$. Now we need to express each $f(v_i)$ in the basis $\{w_1, w_2\}$. These are

$$\begin{aligned} f(v_1) &= 10 \cdot w_1 + 25 \cdot w_2, \\ f(v_2) &= 0 \cdot w_1 + 14 \cdot w_2, \\ f(v_3) &= 22 \cdot w_1 + 61 \cdot w_2. \end{aligned}$$

So the desired matrix is

$$\begin{pmatrix} 10 & 0 & 22 \\ 25 & 14 & 61 \end{pmatrix}.$$

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) If a (2×2) matrix has all eigenvalues equal to 1, then it must be equal to the (2×2) identity matrix.

   (1) True. (2) **False**.

   (b) (5 points) If $A = SDS^{-1}$, then $A^3 = S^3 D^3 S^{-3}$.

   (1) True. (2) **False**.

   (c) (5 points) The product of two diagonal matrices is again diagonal.

   (1) **True**. (2) False.

   (d) (5 points) For any (2×2) matrices A, B, $e^{A+B} = e^A \cdot e^B$.

   (1) True. (2) **False**.

   (e) (5 points) Consider a linear function $f : \mathbb{R}^{24} \to \mathbb{R}^{24}$ such that $\dim(\text{im}(f)) = 21$. Then f cannot be injective.

   (1) **True**. (2) False.