MAT 67: PROBLEM SET 3

DUE TO FRIDAY MAY 10 2024

Abstract. Solutions were typed by TA Scroggin, please contact *tmscroggin – at – ucdavis.edu* for any comments.

Purpose: The goal of this assignment is to acquire the necessary skills to work with linear maps, bases and matrices. These were discussed during the fifth week of the course and are covered in Chapter 6 and Appendix A of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the Office Hours offered by the Professor and the TA. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)

Problem 1. Let $V = W = \mathbb{R}^3$ and let $f : V \to W$ be a linear map. Consider the basis $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$, where

$$v_1 = (1, 0, -2), \quad v_2 = (3, 4, 0), \quad v_3 = (1, -1, 2),$$

$$w_1 = (1, 0, 1), \quad w_2 = (1, 1, 1), \quad w_3 = (0, 0, 1).$$

Suppose that $f(v_i) = w_i$ for $1 \leq i \leq 3$.

(1) Find the vectors $f(5, 3, 0)$, $f(5, 9, -2)$ and $f(1, 0, 0)$.

(2) Find the numbers $a_{ij} \in \mathbb{R}$ such that

$$f(x_1, x_2, x_3) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3, a_{31}x_1 + a_{32}x_2 + a_{33}x_3).$$

Solution.

(1) We use the linearity of f to determine the values of the vectors. Since we are given the where the basis vectors v_i map under f, we write the vectors as a linear combination of the basis vectors v_i. We find that

$$\begin{aligned} (5,3,0) &= 1 \cdot (1,0,-2) + 1 \cdot (3,4,0) + 1 \cdot (1,-1,2) \\ (5,9,-2) &= 0 \cdot (1,0,-2) + 2 \cdot (3,4,0) - 1 \cdot (1,-1,2) \\ (1,0,0) &= \frac{4}{11} \cdot (1,0,-2) + \frac{1}{11} \cdot (3,4,0) + \frac{4}{11} \cdot (1,-1,2) \end{aligned}$$

Using the linearity of f, we find that

$$\begin{aligned} f(5,3,0) &= 1 \cdot f(1,0,-2) + 1 \cdot f(3,4,0) + 1 \cdot f(1,-1,2) \\ &= 1 \cdot (1,0,1) + 1 \cdot (1,1,1) + 1 \cdot (0,0,1) \\ &= \boxed{(2,1,3)}, \\ f(5,9,-2) &= 0 \cdot f(1,0,-2) + 2 \cdot f(3,4,0) - 1 \cdot f(1,-1,2) \\ &= 0 \cdot (1,0,1) + 2 \cdot (1,1,1) - 1 \cdot (0,0,1) \\ &= \boxed{(2,2,1)}, \\ f(1,0,0) &= \frac{4}{11} \cdot f(1,0,-2) + \frac{1}{11} \cdot f(3,4,0) + \frac{4}{11} \cdot f(1,-1,2) \\ &= \frac{4}{11} \cdot (1,0,1) + \frac{1}{11} \cdot (1,1,1) + \frac{4}{11} \cdot (0,0,1) \\ &= \boxed{\left(\frac{5}{11}, \frac{1}{11}, \frac{9}{11}\right)}. \end{aligned}$$

(2) We use the values of the vectors in part (1) to find the numbers $a_{ij} \in \mathbb{R}$. We see that

$$f(5,3,0) = (5a_{11} + 3a_{12}x_2, 5a_{21} + 3a_{22}, 5a_{31} + 3a_{32}) = (2,1,3)$$
$$f(5,9,-2) = (5a_{11} + 9a_{12} - 2a_{13}, 5a_{21} + 9a_{22} - 2a_{23}, 5a_{31} + 9a_{32} - 2a_{33}) = (2,2,1)$$
$$f(1,0,0) = (a_{11}, a_{21}, a_{31}) = \left(\frac{5}{11}, \frac{1}{11}, \frac{9}{11}\right)$$

Which allows us to write three systems of equations

$$\begin{cases} 5a_{11} + 3a_{12}x_2 = 2 \\ 5a_{11} + 9a_{12} - 2a_{13} = 2 \\ a_{11} = \frac{5}{11} \end{cases} \quad \begin{cases} 5a_{21} + 3a_{22} = 1 \\ 5a_{21} + 9a_{22} - 2a_{23} = 2 \\ a_{21} = \frac{1}{11} \end{cases} \quad \begin{cases} 5a_{31} + 3a_{32} = 3 \\ 5a_{31} + 9a_{32} - 2a_{33} = 1 \\ a_{31} = \frac{9}{11} \end{cases}$$

Using your favorite method to solve the systems of equations we find that

$$a_{11} = \frac{5}{11},\ a_{12} = -\frac{1}{11},\ a_{13} = -\frac{3}{11},\ a_{21} = \frac{1}{11},\ a_{22} = \frac{2}{11},$$
$$a_{23} = \frac{1}{22},\ a_{31} = \frac{9}{11},\ a_{32} = -\frac{4}{11},\ a_{33} = -\frac{1}{11}.$$

Therefore, the matrix for f with the given basis is

$$A_f = \begin{pmatrix} 5/11 & -1/11 & -3/11 \\ 1/11 & 2/11 & 1/22 \\ 9/11 & -4/11 & -1/11 \end{pmatrix}$$

□

Problem 2. Consider $V = \mathbb{R}^3, W = \mathbb{R}^2, Z = \mathbb{R}^2$ and the following two linear maps $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$:

$$f(x_1,x_2,x_3) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & -5 \\ -4 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad g(y_1,y_2,y_3) = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Each item is worth 5 points. Solve the following parts:

(1) Compute $f(1,3,-1)$ and $g(2,5,0)$.

(2) Find a matrix expression for the composition $g \circ f : \mathbb{R}^3 \to \mathbb{R}^2$.

(3) Find bases for the nullspaces $\ker(f)$, $\ker(g)$ and $\ker(g \circ f)$.

(4) Find bases for the ranges $im(f)$, $im(g)$ and $im(g \circ f)$.

Solution.

(1)

$$f(1,3,-1) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & -5 \\ -4 & 3 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 8 \\ 20 \\ 12 \end{pmatrix}},$$

$$g(2,5,0) = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 40 \end{pmatrix}.}$$

(2)

$$g \circ f = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & -5 \\ -4 & 3 & -7 \end{pmatrix} = \boxed{\begin{pmatrix} -11 & 11 & -22 \\ 5 & 40 & -35 \end{pmatrix}.}$$

(3) To find the bases for $\ker(f)$, we solve the system of equations associated to

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & -5 \\ -4 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, we get the three equations

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 5x_2 - 5x_3 = 0 \\ -4x_1 + 3x_2 - 7x_3 = 0 \end{cases}.$$

We see that $x_3 = x_2$ from equation (2), which allows us to solve for $x = -x_2$ in equation (1) and by equation (3) we solve for $x_2 = 0$. Therefore, $\boxed{\ker(f) = \{0\}}$.

To solve for the bases of $\ker(g)$, we solve the systems of equations

$$\begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Which is equivalent to

$$\begin{cases} y_1 + 3y_3 = 0 \\ 5y_1 + 6y_2 = 0 \end{cases}$$

From equation (1) we see that $y_3 = -\frac{1}{3}y_1$ and from equation (2) we see that $y_2 = -\frac{5}{6}y_1$. So all vectors in $\ker g$ are of the form

$$\left(y_1, -\frac{5}{6}y_1, -\frac{1}{3}y_1\right) \sim (6y_1, -5y_1, -2y_1)$$

by clearing the fractions. Therefore, a basis for $\ker g$ is $(6, -5, -2)$, i.e., $\boxed{\ker g = \text{span}(6, -5, -2)}$.

Finally we find the basis for $\ker g \circ f$ by solving the system of equations associated to

$$\begin{pmatrix} -11 & 11 & -22 \\ 5 & 40 & -35 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Which is the system of equations

$$\begin{cases} -11x_1 + 11x_2 - 22x_3 = 0 \\ 5x_1 + 40x_2 - 35x_3 = 0 \end{cases} \implies \begin{cases} -x_1 + x_2 - 2x_3 = 0 \\ x_1 + 8x_2 - 7x_3 = 0 \end{cases}.$$

We solve for x_1 in both equations and set them equal to each other and find that $x_2 - 2x_3 = -8x_2 + 7x_3$, then $x_2 = x_3$. Therefore, $x_1 = x_3 - 2x_3 = -x_3$, so

$$\boxed{\ker g \circ f = \operatorname{span}(-1, 1, 1).}$$

(4) Here, we may look at the column vectors of the matrices and determine which of the column vectors are linearly independent. For f, we have that $(-1, -5, -7) = (1, 0, -4) - (2, 5, 3)$, so $\boxed{\operatorname{Im} f = \operatorname{span}\{(1, 0, -4), (2, 5, 3)\}}$.
For g, we see that $(1, 5) = \frac{1}{3}(3, 0) + \frac{1}{6}(0, 6)$, so $\boxed{\operatorname{Im} g = \operatorname{span}\{(3, 0), (0, 6)\}}$. And for $g \circ f$, we see that $(-22, -35) = (-11, 5) - (11, 40)$, so $\boxed{\operatorname{Im} g \circ f = \operatorname{span}\{(-11, 5), (11, 40)\}}$.

□

Problem 3. From the textbook. Solve the Exercises (1), (2) and (6) in Page 86 (End of Chapter 6). The first two count 8 points and the last one 9 points.

(1) Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.
  (a) Show that T is linear.
  (b) Show that T is surjective.
  (c) Find $\dim(null(T))$.
  (d) Find the matrix for T with respect to the canonical basis of $\mathbb{R}^2$.
  (e) Find the matrix for T with respect to the canonical basis for the domain $\mathbb{R}^2$ and the basis $((1, 1), (1, -1))$ for the target space $\mathbb{R}^2$.
  (f) Show that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x + y, x + 1)$ is not linear.

Solution. (a) This is a straightforward check:

$$\begin{aligned} T(x + x', y + y') &= (x + x' + y + y', x + x') = (x + y, x) + (x' + y', x') \\ &= T(x, y) + T(x' + y', x'), \\ T(c \cdot (x, y)) &= T(c \cdot x, c \cdot y) = (c \cdot x + c \cdot y, c \cdot x) = c \cdot (x + y, x) \\ &= c \cdot T(x, y). \end{aligned}$$

(b) The map $T(x, y) = (x + y, x) = (x, x) + (y, 0)$, since $\{(1, 1), (1, 0)\}$ are linearly independent, then the image of the map is the span$((1, 1), (1, 0))$. Therefore, the image of T is $\mathbb{R}^2$ and the map is surjective.

Alternatively, from previous methods, let $x = b, y = a - b$, then for all $(a, b) \in \mathbb{R}^2$ there exists an $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (a, b)$, i.e., T is surjective.

(c) We want to find all $(x, y) \in \mathbb{R}^2$ such that $T(x, y) = (x + y, x) = (0, 0)$. Solving for x, y we see that $y = -x$ and $x = 0$, therefore, $\ker T = \{0\}$ and $\dim \ker T = 0$.

(d) The canonical basis for $\mathbb{R}^2$ is given by $e_1 = (1, 0), e_2 = (0, 1)$. The matrix of T is given by $T(e_1), T(e_2)$ written as column vectors. We see that

$T(e_1) = (1,1)$ and $T(e_2) = (1,0)$, so the matrix for T with respect to the canonical basis of $\mathbb{R}^2$ is given by

$$\boxed{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}.$$

(e) First, we compute the canonical basis vectors under the map T and then express these vectors in terms of the target basis $((1,1),(1,-1))$.

$$T(1,0) = (1,1) = 1 \cdot (1,1) + 0 \cdot (1,-1) = a_{11}(1,1) + a_{21}(1,-1)$$
$$T(0,1) = (1,0) = \frac{1}{2} \cdot (1,1) + \frac{1}{2} \cdot (1,-1) = a_{12}(1,1) + a_{22}(1,-1)$$

Therefore, the matrix T is given by

$$\boxed{\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}}.$$

(f) We can clearly see that the map F is not linear since

$$F(x+x', y+y') = (x+x'+y+y', x+x'+1) \neq (x+y, x+1)+(x'+y', x'+1) = F(x,y)+F(x',y').$$

□

(2) Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}, \text{ for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

(a) Show that T is surjective.
(b) Find $\dim(null(T))$.
(c) Find the matrix for T with respect to the canonical basis of $\mathbb{R}^2$.
(d) Show that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = (x+y, x+1)$ is not linear.

Solution. (a) The map $T(x,y) = (y,-x) = (y,0)+(0,-x)$. Since $\{(1,0),(0,-1)\}$ are linearly independent, then the image of the map is $\text{span}((1,0),(0,-1)) = \mathbb{R}^2$, so T is a surjective map.

Alternatively, let $x = -b, y = a$, then for all $(a,b) \in \mathbb{R}^2$ there exists an $(x,y) \in \mathbb{R}^2$ such that $T(x,y) = (a,b)$.

(b) We want to find all $(x,y) \in \mathbb{R}^2$ such that $T(x,y) = (0,0)$. Then $x = y = 0$ and $\ker T = \{0\}$, so $\boxed{\dim(null(T)) = 0}$.

(c) We see that $T(e_1) = T(1,0) = (0,-1)$ and $T(e_2) = T(0,1) = (1,0)$, so the matrix for T is given by

$$\boxed{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}.$$

(d) We can plainly see that the map F is not linear since for $c \neq 1$

$$c \cdot F(x,y) = (cx+cy, cx+c) \neq (cx+cy, cx+1) = F(c \cdot (x,y)).$$

□

(3) Show that no linear map $T : \mathbb{F}^5 \to \mathbb{F}^2$ can have as its null space the set

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 | x_1 = 3x_2, x_3 = x_4 = x_5\}.$$

Proof. By the dimension formula we see that

$$\dim \mathbb{R}^5 = \dim \ker T + \dim ImT$$

Since the dimension of the codomain $\mathbb{F}^2$ is 2 dimensional, then $0 \leq \dim ImT \leq 2$ which implies that $3 \leq \dim \ker T \leq 5$. However, the subset is defined

$$\{(3x_2, x_2, x_3, x_3, x_3) \in \mathbb{F}^5 : x_2, x_3 \in \mathbb{F}\}$$

and is only 2-dimensional; therefore, the subset cannot be defined as the kernel for the map T since $2 < 3$.

□

Problem 4. From the textbook. Solve the Proof-Writing Exercises (1), (2), (4) and (7) in Page 83 (End of Chapter 6). The first three count 7 points and the last one 4 points

(1) Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional, and let U be any subspace of V . Given a linear map $S \in \mathcal{L}(U, W)$, prove that there exists a linear map $T \in \mathcal{L}(V, W)$ such that, for every $u \in U$, $S(u) = T(u)$.

Proof. Let U be a subspace of V and let $V = \text{span}(v_1, \ldots, v_k, \ldots, v_n)$ and $U = \text{span}(v_1, \ldots, v_k)$. We define the map $T \in \mathcal{L}(V, W)$ using the basis vectors of V, then

$$T(v_i) = \begin{cases} S(v_i) & \text{if } 1 \leq i \leq k \\ 0 & \text{if } k+1 \leq i \leq n \end{cases}$$

Under this map we see that for any vector $u \in U$ expressed as $u = a_1v_1 + \cdots + a_kv_k$ that

$$\begin{aligned} T(u) &= T(a_1v_1 + \cdots + a_kv_k) = a_1T(v_1) + \cdots + a_kT(v_k) \\ &= a_1S(v_1) + \cdots + a_kS(v_k) = S(a_1v_1 + \cdots + a_kv_k) \\ &= S(u) \end{aligned}$$

Therefore, given a linear map $S \in \mathcal{U}, \mathcal{W}$ there exists a linear map $T \in \mathcal{L}(V, W)$ where $T(u) = S(U)$ for every $u \in U$. □

(2) Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is injective. Given a linearly independent list $(v_1, \ldots, v_n)$ of vectors in V, prove that the list $(T(v_1), \ldots, T(v_n))$ is linearly independent in W.

Proof. Suppose $(v_1, \ldots, v_n)$ is a set linearly independent vectors in V, then the only solution to

$$a_1v_1 + \cdots + a_nv_n = 0$$

is the trivial solution where $a_i = 0$ for all $1 \leq i \leq n$. Since $T \in \mathcal{L}(V, W)$ is an injective linear map then $\ker T = \{0\}$, i.e., if $T(v) = 0$ then $v = 0$. Let $v \in V$ where $v = a_1v_1 + \cdots + a_nv_n = 0$, then by the linearity of T we see that

$$\begin{aligned} 0 = T(v) &= T(a_1v_1 + \cdots + a_nv_n) \\ &= a_1T(v_1) + \cdots + a_nT(v_n) \end{aligned}$$

Here $a_i = 0$ for all $1 \leq i \leq n$ as above, therefore, $(T(v_1), \ldots, T(v_n))$ is a linearly independent set of vectors in W. □

(3) Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is surjective. Given a spanning list $(v_1, \ldots, v_n)$ for V , prove that

$$\operatorname{span}(T(v_1), \ldots, T(v_n)) = W.$$

Proof. We want to show that for any vector $w \in W$ that we may write

$$w = a_1 T(v_1) + \cdots + a_n T(V_n).$$

We are given that $T \in \mathcal{L}(V, W)$ is surjective, therefore, for all $w \in W$ there exists a $v \in V$ where $T(v) = w$. Since $(v_1, \ldots, v_n)$ is a spanning set of V then we may write any vector $v \in V$ as

$$v = a_1 v_1 + \cdots + a_n v_n.$$

By the linearity of T we find that

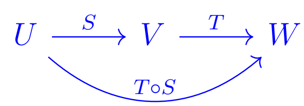
$$\begin{aligned} w = T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n). \end{aligned}$$

Therefore, any $w \in W$ can be expressed as a linear combination of $(T(v_1), \ldots, T(v_n))$, i.e., $\operatorname{span}(T(v_1), \ldots, T(v_n)) = W$. ☐

(4) Let U, V, and W be finite-dimensional vector spaces over $\mathbb{F}$ with $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that $\dim(null(T \circ S)) \leq \dim(null(T)) + \dim(null(S))$.

Proof. I will use a minor change of notation, I will use ker instead of *null*, and $F(X)$ to denote the image of set X under some arbitrary map F instead of ImF which was used in discussion.

The maps are represented by the picture:



As a quick note, if $u \in U$ is an element in $\ker(T \circ S)$, i.e., $(T \circ S)(u) = 0$, then either $S(u) = 0$ ($u \in \ker S$), or $S(u) \neq 0$ and $T(S(u)) = 0$ ($S(u) \in \ker T$).

Since $\ker T \circ S \subseteq U$, then

$$\dim \ker T \circ S \leq \dim U = \dim \ker S + \dim S(U).$$

Also, since $S(U) \subseteq V$, then

$$\dim S(U) \leq \dim V = \dim \ker T + \dim T(V).$$

By combining these inequalities, we see that

$$\dim \ker T \circ S \leq \dim \ker S + \dim \ker T + \dim T(V).$$

Now, we want to show that we can remove the additional term $\dim T(V)$ in the inequality. Since $\ker T \circ S \subseteq U$ we may restrict the map S to the subspace $\ker T \circ S$, which provides an additional dimension formula

$$\dim(\ker T \circ S) = \dim \ker S + \dim S(\ker T \circ S)$$

Given that $S(\ker T \circ S) \subseteq V$, i.e., we can apply the map T to the subspace $S(\ker T \circ S)$, then

$$\dim S(\ker T \circ S) \leq \dim \ker T + \dim T(S(\ker T \circ S))$$

Finally, we combine the last two inequalities to get

$$\dim(\ker T \circ S) \leq \dim \ker S + \dim \ker T + \dim T(S(\ker T \circ S)).$$

Note that $T(S(\ker T \circ S)) = 0$ by definition, so $\dim T(S(\ker T \circ S)) = 0$. Therefore,

$$\dim(\ker T \circ S) \leq \dim \ker S + \dim \ker T.$$

□