MAT 67: PROBLEM SET 4

DUE TO FRIDAY MAY 17 2024

Abstract. This problem set corresponds to the sixth week of the course MAT-67 Spring 2024. It is due Friday May 17 at 9:00am submitted via Gradescope.

Purpose: The goal of this assignment is to acquire the necessary skills to work with determinants. These were discussed during the sixth week of the course and are covered in Chapter 8 of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the Office Hours offered by the Professor and the TA. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)

Problem 1. Decide whether the following sets of vectors are linearly independent:

(1) Let $v_1, v_2, v_3 \in \mathbb{R}^3$ where
$$v_1 = (1, 0, -2), \quad v_2 = (3, 4, 0), \quad v_3 = (1, -1, 2),$$

(2) Let $v_1, v_2, v_3 \in \mathbb{R}^5$ where
$$v_1 = (1, 0, 1, 0, -2), \quad v_2 = (0, 7, -3, 4, 0), \quad v_3 = (0, 0, 1, -1, 2),$$

(3) Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^5$ where
$$\begin{aligned} v_1 &= (1, 0, 1, 0, -2), \quad & v_2 &= (0, 7, -3, 4, 0), \\ v_3 &= (2, 7, 0, 3, -2), \quad & v_4 &= (0, 0, 1, -1, 2). \end{aligned}$$

Problem 2. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be linear maps.

(1) Show that $\det(g) \cdot \det(f) = \det(g \circ f)$.

(2) By definition, a linear map $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is *invertible* if there exists a linear map $h^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $h \circ h^{-1} = \mathrm{Id} = h^{-1} \circ h$, where Id is the identity.

Show that h is invertible if and only if $\det(h) \neq 0$.

(3) Decide whether the map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by the matrix
$$A_f = \begin{pmatrix} -2 & 0 & -1 & -1 \\ 1 & 5 & -5 & 0 \\ -3 & 3 & -7 & 4 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$
is invertible.

Problem 3. From the textbook. Solve the Calculation Exercises (4) and (5) in Page 118 (End of Chapter 8). The first counts 10 points and the second 15 points.

Problem 4. From the textbook. Solve the Proof-Writing Exercises (1), (3), and (4) in Page 118 (End of Chapter 8). The first one counts 15 points and the second and last one 5 points.