MAT 141: PROBLEM SET 2

DUE TO FRIDAY JAN 24 AT 10:00AM

Abstract. This is the second problem set for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. It was posted online on Friday Jan 17 and is due Friday Jan 24 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice problems on isometries of the Euclidean Plane $\mathbb{R}^2$. In particular, we would like to become familiar with the classification of planar isometries, including rotations, translations and glide reflections.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Geometry of Surfaces" by J. Stillwell.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry and $A, B, C \in \mathbb{R}^2$ three non-collinear points. Suppose that $f(A) \neq A, f(B) \neq B$ and $f(C) \neq C$. Show that f is the product of one, two or three reflections.

Comment: This is one of the cases of our Classification Theorem for Isometries of the Euclidean Plane. Thus, you are not allowed to use the Theorem.

Problem 2. Given an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, an invariant line $l \subseteq \mathbb{R}^2$ is a line that gets mapped by f onto itself, i.e. $f(L) = L$. Note that this does **not** mean that the points $p \in L$ are fixed.

(a) Use the Classification Theorem of Isometries in the Euclidean Plane to show that an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ has exactly one of the following:

(i) A line of fixed points,
(ii) A single fixed point,
(iii) No fixed points, and a parallel family of invariant lines,
(iv) No fixed points, and a single invariant line.

Remark: In particular, it is possible to **define** *points and lines starting from the group of isometries itself. This is beginning of the Erlangen program, a theory initiated by F. Klein in 1872, whose tenet is the development of geometries in terms of their isometries.*

(b) In each of the four cases in Part 2.(a), describe the isometry f as a product of one, two or three reflections along lines.

Hint: you shall need to describe the relative position of these lines.

Problem 3. (20 pts) (**Glide reflections**) A glide reflection is an plane isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form $t_{(\alpha,\beta)} \circ \overline{r}_L$ with the translation vector $(\alpha, \beta) \in \mathbb{R}^2$ parallel to the reflection line L.

(a) Let $t_{(\alpha,\beta)} \circ \overline{r}_L$ be a glide reflection. Show that $t_{(\alpha,\beta)} \circ \overline{r}_L = \overline{r}_L \circ t_{(\alpha,\beta)}$.

(b) Give an example of a point $(\gamma, \delta) \in \mathbb{R}^2$ and a line $M \subseteq \mathbb{R}^2$ such that

$$t_{(\gamma,\delta)} \circ \overline{r}_M \neq \overline{r}_M \circ t_{(\gamma,\delta)}.$$

(c) Let $(\alpha, \beta) \in \mathbb{R}^2$ and $L \subseteq \mathbb{R}^2$ a line. Suppose that

$$t_{(\alpha,\beta)} \circ \overline{r}_L = \overline{r}_L \circ t_{(\alpha,\beta)}.$$

Show that $(\alpha, \beta) \in \mathbb{R}^2$ parallel to L.

Note: Thus, glide reflections can also be defined as those compositions of a reflection and a translation which commute.

(d) Consider the rectangular box $B = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 2 \leq y \leq 4\} \subseteq \mathbb{R}^2$, and let $f = t_{(4,0)} \circ \overline{r}$ be a glide reflection. Draw the five set

$$B, f(B), f^2(B), f^3(B), f^4(B), f^5(B),$$

defined by the iterated images of the box B under the isometry f.

(e) Let $L, M, N \subseteq \mathbb{R}^2$ be three lines. Show that a product $\overline{r}_N \overline{r}_M \overline{r}_L$ of three reflection is a glide reflection.

Hint: It might be helpful to study the different cases depending on the relative positions of the lines $L, M, N \subseteq \mathbb{R}^2$.

Problem 4. (20 pts) The goal of this exercise is to complete the following table:

	Reflection $\overline{r}_L$	Translation $t_{(\alpha,\beta)}$	Rotation $R_{\theta,P}$	Glide reflection
Reflection $\overline{r}_M$				.
Translation $t_{(\gamma,\delta)}$				
Rotation $R_{\phi,Q}$				
Glide Reflection				

The table is completed as follows. At a given entry, we want to describe the type of isometry $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is obtained by composing an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the type indicated by its row with an isometry $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the type indicated by its column. There are a total of four types: reflections, translations, rotations and glide-reflections. There can be more than one type per entry.

In general, we will include reflections $\overline{r}_L$ within the set of glide reflections. Just for the purpose of this problem, *glide reflection* refers to a glide reflection which is not a reflection.

(a) Show that $\overline{r}_M \overline{r}_L$ is either a rotation or a translation. What is the geometric position between M and L if $\overline{r}_M \overline{r}_L$ is a translation ?

(b) Show that the composition of a glide reflection with a reflection is a rotation or a translation.

(c) Complete the table above.

(d) The order in which we compose isometries can matter. Show that a rotation and a reflection do not necessarily commute.

(e) Discuss whether glide-reflections commute with reflections, translations and rotations.

Problem 5. (20 pts) Let $T \subseteq \mathbb{R}^2$ be the equilateral triangle centered at the origin.

(a) Show that there are exactly *six* isometries $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which verify $f(T) = T$. Let us call them $s_1, s_2, s_3, s_4, s_5, s_6$.

(b) Explain why the composition $s_i \circ s_j$, for any $1 \leq i, j \leq 6$, must be of the form s_k, for some $1 \leq k \leq 6$.

(c) Complete the table below according to this product rule: in entry ij in the table is s_k if $s_j \circ s_i = s_k$. In particular, explain why the set of isometries which preserve T form a group G_T.

	s_1	s_2	s_3	s_4	s_5	s_6
s_1						
s_2						
s_3						
s_4						
s_5						
s_6						

(d) Is the group G_T commutative, i.e. $s_i \circ s_j = s_j \circ s_i$, for all $1 \leq i, j \leq 6$?

(e) Consider the set $I = \{1, 2, 3\}$ with three elements. Show that there are exactly six bijections $F : I \longrightarrow I$. Let us call them $F_1, F_2, F_3, F_4, F_5, F_6$.

(f) Complete the following table, where in the (ij) entry we write the bijection F_k which corresponds to the composition $F_j \circ F_i$.

	F_1	F_2	F_3	F_4	F_5	F_6
F_1						
F_2						
F_3						
F_4						
F_5						
F_6						

(g) Show that there is a relabeling of $s_1, s_2, s_3, s_4, s_5, s_6$ into $F_1, F_2, F_3, F_4, F_5, F_6$ such that the two tables in Part 2.(c) and Part 2.(f) above coincide.

This proves that the group of isometries of the regular triangle is the same as the group of bijections of three elements.

(h) Let $S \subseteq \mathbb{R}^2$ the square with vertices $(1,1), (1,-1), (-1,1), (-1,-1) \in \mathbb{R}^2$. How many isometries $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are there such that $f(S) = S$?

Problem 6. (20 pts) For each of the ten sentences below, justify whether they are **true** or **false**. If true, you must provide a proof, if false you must provide a counter-example.

(a) Planar Isometries preserve angles. That is, let $O, P, Q \in \mathbb{R}^2$ be points and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ an isometry. Then the angle between the vectors $\vec{OP}$ and $\vec{OQ}$ equals the angle between the vectors $\overrightarrow{f(O)f(P)}$ and $\overrightarrow{f(O)f(Q)}$.

(b) The set of rotations $R_{\theta,P} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ form a group inside the isometry group of the Euclidean plane.

(c) The set of translations $t_{(\alpha,\beta)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ form a group inside the isometry group of the Euclidean plane.

(d) Let us call an isometry *orientation-preserving* if it is the product of two reflections. The set of orientation-preserving isometries is a group.

(e) Let us call an isometry *orientation-reversing* if it is the product of one or three reflections. The set of orientation-reversing isometries is a group.

(f) Suppose that $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are isometries and $A, B, C \in \mathbb{R}^2$ are three points. If $f(A) = g(A), f(B) = g(B)$ and $f(C) = g(C)$, then $f = g$.

(g) Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be isometries that fix *all* points of the same line $L \subseteq \mathbb{R}^2$. Then it must be that $f = g$.

(h) Let T be the triangle in Problem 5, and $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be isometries such that the vertices of the triangle $f(T)$ coincide with the vertices of the triangle $g(T)$. Then $f = g$.

(i) Let T be the triangle in Problem 5, and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry such that $f(P) = P$ for $P \in T$. Then $f(Q) = Q$ for all $Q \in \mathbb{R}^2$.

(j) Let $A, B, C, D \in \mathbb{R}^2$ be four points. There always exists a point $P \in \mathbb{R}^2$ such that $d(P, A) = d(P, B) = d(P, C) = d(P, D)$.

Problem 7. (20 pts) In this problem we will explore *wall-paper* tilings of the Euclidean plane. There is a classification of *all* possible regular tiling of the plane, resulting in 17 isometry subgroup of the group of isometries of $\mathbb{R}^2$. In this exercise we will study some of them. Each of the patterns displayed in the Figures should be understood as extending infinitely all over the Euclidean Plane $\mathbb{R}^2$.

(a) Explore whether the pattern in Figure 1 admits any translational symmetries, i.e. it is invariant under translations in the Euclidean plane. Is it invariant under any reflection ?

(b) Explore whether the pattern in Figure 2 admits any rotational symmetries, i.e. whether it is invariant under certain rotations of the Euclidean plane. Are there any reflection which preserve this pattern ?

(c) Find all translations which leave the pattern in Figure 3 invariant.

(d) Compare the isometries which preserve the brick pattern in Figure 4 with the isometries that preserve the tile pattern in Figure 5.

(e) (Bonus - Extra Credit) Explore the isometries of the patterns in Figures 6, 7, 8 and 9.

FIGURE 1. The pattern in the Patio de los Arrayanes in *La Alhambra*, Spain.

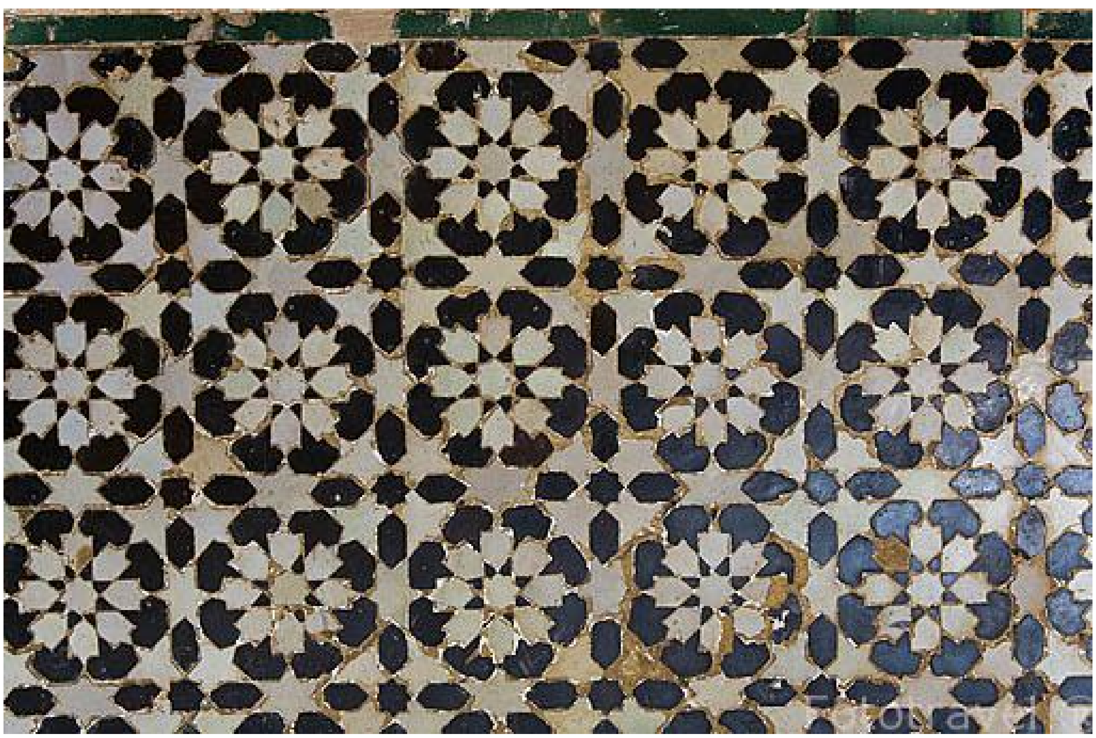

FIGURE 2. The pattern in La Torre de las Infantas in *La Alhambra*, Spain.

FIGURE 3. A tiling of the plane made of fish and birds.

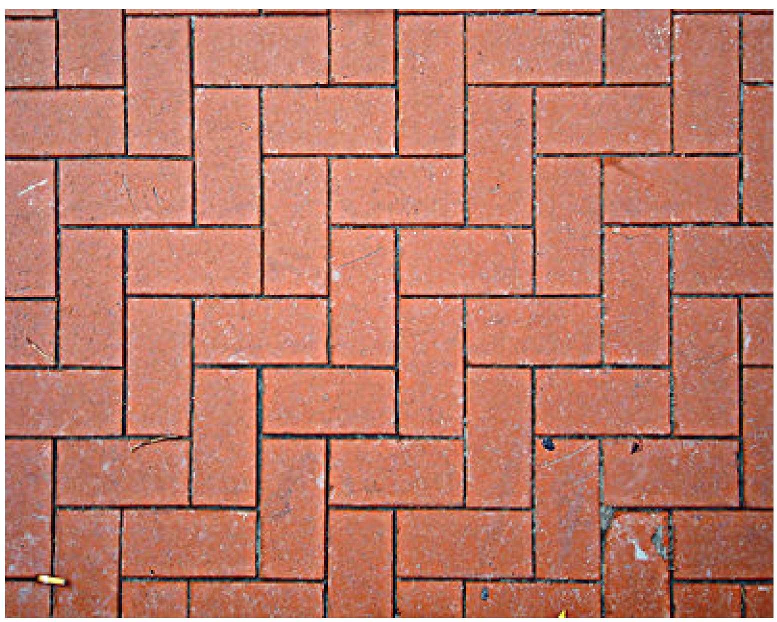

FIGURE 4. First Brick Tiling.

Figure 5. Second Brick Tiling.

Figure 6. M.C. Escher Birds.

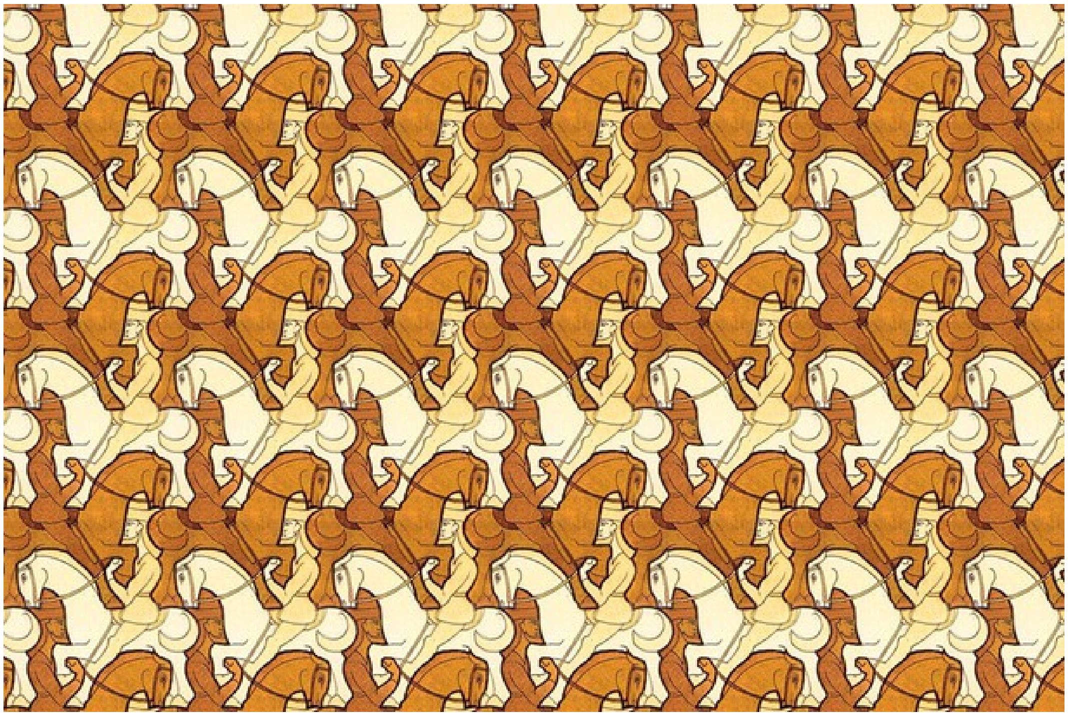

FIGURE 7. General in Horse Tiling.

FIGURE 8. M.C. Escher's fish.

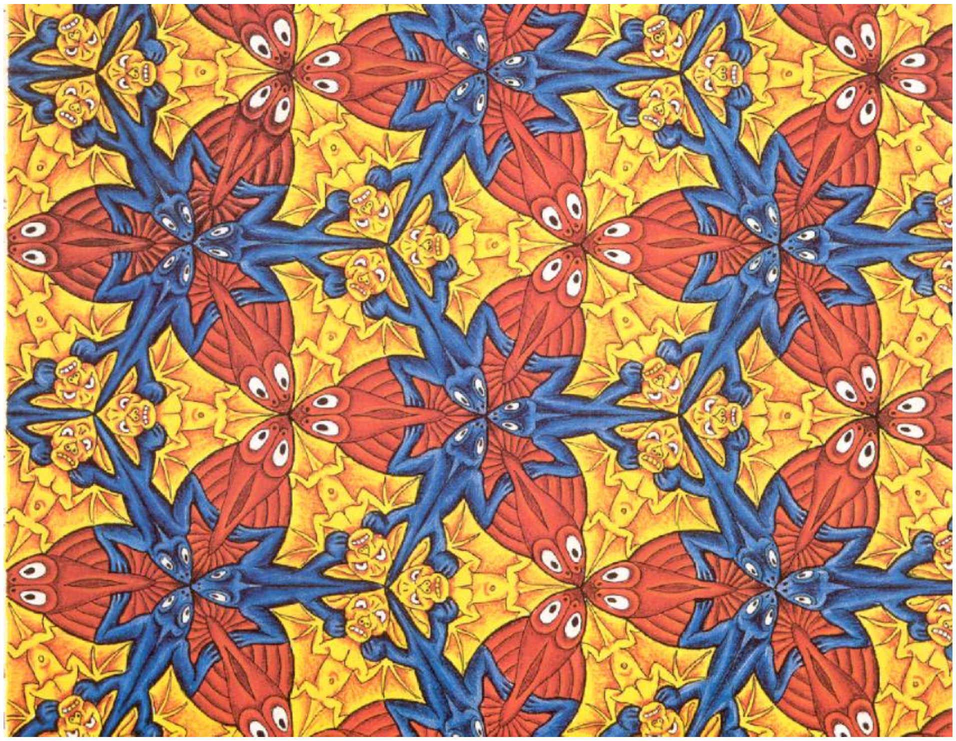

FIGURE 9. M.C. Escher's Artwork.