SOLUTIONS TO PROBLEM SET 1

MAT 141

Abstract. These are the solutions to Problem Set 2 for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. The problems were posted online on Friday Jan 17 and due Friday Jan 24 at 10:00am.

Problem 1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry and $A, B, C \in \mathbb{R}^2$ three non-collinear points. Suppose that $f(A) \neq A, f(B) \neq B$ and $f(C) \neq C$. Show that f is the product of one, two or three reflections.

Comment: This is one of the cases of our Classification Theorem for Isometries of the Euclidean Plane. Thus, you are not allowed to use the Theorem.

Solution.

First, perform the reflection $\overline{r}_a$ in the line of points equidistant from A and $f(A)$. Then $\overline{r}_a(A) = f(A)$. If $\overline{r}_a = f$, then f is a reflection, and we are done. Otherwise, without loss of generality, $\overline{r}_a(B) \neq f(B)$. So perform the reflection $\overline{r}_b$ in the line of points equidistant from $\overline{r}_a(B)$ and $f(B)$, which exchanges these points. Since

$$d(f(A), \overline{r}_a(B)) = d(\overline{r}_a(A), \overline{r}_a(B)) = d(f(A), f(B)),$$

we see that $f(A)$ is equidistant from $\overline{r}_a(B)$ and $f(B)$, so it is fixed by $\overline{r}_b$. Therefore, we have $\overline{r}_b\overline{r}_a(A) = f(A)$, and $\overline{r}_b\overline{r}_a(B) = f(B)$. If $\overline{r}_b\overline{r}_a = f$, then f is the product of two reflections, and we are done. Otherwise $\overline{r}_b\overline{r}_a(C) \neq C$, so perform the reflection $\overline{r}_c$ in the line of points equidistant from $\overline{r}_b\overline{r}_a(C)$ and $f(C)$, exchanging these points. We have

$$d(f(A), \overline{r}_b\overline{r}_a(C)) = d(\overline{r}_b\overline{r}_a(A), \overline{r}_b\overline{r}_a(C)) = d(f(A), f(C)),$$

so $f(A)$ is equidistant from $\overline{r}_b\overline{r}_a(C)$ and $f(C)$, and hence fixed by $\overline{r}_c$. The same is true for $f(B)$. We conclude that $\overline{r}_c\overline{r}_b\overline{r}_a(A) = f(A)$, $\overline{r}_c\overline{r}_b\overline{r}_a(B) = f(B)$, and $\overline{r}_c\overline{r}_b\overline{r}_a(C) = f(C)$, so $\overline{r}_c\overline{r}_b\overline{r}_a = f$, and f is the product of three reflections.

Problem 2. Given an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, an invariant line $l \subseteq \mathbb{R}^2$ is a line that gets mapped by f onto itself, i.e. $f(L) = L$. Note that this does **not** mean that the points $p \in L$ are fixed.

(a) Use the Classification Theorem of Isometries in the Euclidean Plane to show that an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ has exactly one of the following:

(i) A line of fixed points,
(ii) A single fixed point,
(iii) No fixed points, and a parallel family of invariant lines,
(iv) No fixed points, and a single invariant line.

Remark: In particular, it is possible to **define** *points and lines starting from the group of isometries itself. This is beginning of the Erlangen program, a theory initiated by F. Klein in 1872, whose tenet is the development of geometries in terms of their isometries.*

(b) In each of the four cases in Part 2.(a), describe the isometry f as a product of one, two or three reflections along lines.

Hint: you shall need to describe the relative position of these lines.

Solution.

(a) It is clear that f cannot have more than one of these properties, so we only need to show that it has one of them. The Classification Theorem of Isometries in the Euclidean Plane says that f is either a rotation, a translation, or a glide reflection. If we dispense with the possibility that f is the identity (so f has property (i)), then these three types of isometries are disjoint (because glide reflections reverse orientation while the others don't, and rotations preserve one point while translations don't).

If f is a rotation, say about the point P, then it fixes only the point P, so f has property (ii).

If f is a translation, then it has no fixed points. In this case, say $f = t_{\alpha,\beta}$. Then, for any $c \in \mathbb{R}$, the line L_c defined by $\alpha y - \beta x = c$ is invariant under f. The family

$$\{L_c \mid c \in \mathbb{R}\}$$

is then an infinite family of parallel invariant lines, so f has property (iii).

Finally, suppose f is a glide reflection. If f is a pure reflection, then it fixes the line in which it reflects, so f has property (i). Otherwise, $f = t_P\overline{r}_L$ for some $P \neq O$ such that the segment PO is parallel to the line L (you hop over a river and then follow the river). Consider first the reflection $\overline{r}_L$. All points on a given side of $\overline{r}_L$ have been flipped to the other side. Performing the translation t_P keeps them on this new side, since they only move parallel to L. Therefore, f does not fix any points not in L. Nor does f fix any points in L, because such points are fixed by $\overline{r}_L$ and then moved downstream by $t_P \neq \mathrm{Id}$. However, this observation shows that L is an *invariant* line under f.

We lastly show that f has no other invariant lines. Suppose M is a line distinct from L. If M crosses L, then $\overline{r}_L$ changes the slope of M, and the subsequent translation leaves that new slope unchanged, so M cannot be invariant under f. If M is parallel to L, then it is totally contained on one side of L, and f moves M to the other side, so it again is not invariant. We conclude that f has property (iv).

(b) In case (i), f is a single reflection or the identity (which is the product of two instances of the reflection through any single line).

In case (ii), f is a rotation $R_{\theta,P}$ for some $\theta \neq 0$. Then $f = \overline{r}_M\overline{r}_L$, where M and L are any two lines which meet at P and have angle $\theta/2$ from L to M.

In case (iii), f is a translation t_P for $P \neq O$. Then $f = \overline{r}_M \overline{r}_L$ where M and L are any two parallel lines perpendicular to the segment PO and separated by a distance $\|P\|/2$ from L to M.

In case (iv), f is a glide reflection $t_P \overline{r}_N$, where $P \neq O$ and the line L is parallel to the segment PO. Then $t_P = \overline{r}_M \overline{r}_L$ can be deconstructed as above, so we have $f = \overline{r}_M \overline{r}_L \overline{r}_N$. Here N is known from the original decomposition, and M and L are any lines perpendicular to L such that M is a distance $\|P\|/2$ from L.

Problem 3. (20 pts) (**Glide reflections**) A glide reflection is an plane isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form $t_{(\alpha,\beta)} \circ \overline{r}_L$ with the translation vector $(\alpha, \beta) \in \mathbb{R}^2$ parallel to the reflection line L.

(a) Let $t_{(\alpha,\beta)} \circ \overline{r}_L$ be a glide reflection. Show that $t_{(\alpha,\beta)} \circ \overline{r}_L = \overline{r}_L \circ t_{(\alpha,\beta)}$.

(b) Give an example of a point $(\gamma, \delta) \in \mathbb{R}^2$ and a line $M \subseteq \mathbb{R}^2$ such that

$$t_{(\gamma,\delta)} \circ \overline{r}_M \neq \overline{r}_M \circ t_{(\gamma,\delta)}.$$

(c) Let $(\alpha, \beta) \in \mathbb{R}^2$ and $L \subseteq \mathbb{R}^2$ a line. Suppose that

$$t_{(\alpha,\beta)} \circ \overline{r}_L = \overline{r}_L \circ t_{(\alpha,\beta)}.$$

Show that $(\alpha, \beta) \in \mathbb{R}^2$ parallel to L.

Note: Thus, glide reflections can also be defined as those compositions of a reflection and a translation which commute.

(d) Consider the rectangular box $B = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 2 \leq y \leq 4\} \subseteq \mathbb{R}^2$, and let $f = t_{(4,0)} \circ \overline{r}$ be a glide reflection. Draw the five set

$$B, f(B), f^2(B), f^3(B), f^4(B), f^5(B),$$

defined by the iterated images of the box B under the isometry f.

(e) Let $L, M, N \subseteq \mathbb{R}^2$ be three lines. Show that a product $\overline{r}_N \overline{r}_M \overline{r}_L$ of three reflection is a glide reflection.

Hint: It might be helpful to study the different cases depending on the relative positions of the lines $L, M, N \subseteq \mathbb{R}^2$.

Solution.

(a) By performing a suitable isometry, we may assume that L is the x-axis, which means $\beta = 0$. We calculate

$$t_{\alpha,0} \circ \overline{r}(x, y) = t_{\alpha,0}(x, -y) = t_{\alpha,0}(x + \alpha, -y),$$

and

$$\overline{r} \circ t_{\alpha,0}(x, y) = \overline{r}(x + \alpha, y) = (x + \alpha, -y),$$

so these isometries agree.

Remark: The key point is that each of $t_{(\alpha,0)}$ and $\overline{r}$ operated on only one of the coordinates, so they had no interaction. For a general glide isometry, we have the same phenomenon, where the two functions operate in perpendicular directions, and thus have no interaction. This can best be seen by breaking (x, y) into its components along a basis in these perpendicular directions, which is essentially changing coordinates, as in the "suitable isometry" referenced above.

(b) If M and (γ, δ) are not parallel, then these maps will not not commute, because they won't be acting in perpendicular directions. The extreme case occurs when M and (γ, δ) are *perpendicular*, so the actions of $t_{(\gamma,\delta)}$ and $\overline{r}_M$ are parallel. Take M to be the x-axis, and set $(\gamma, \delta) = (0, 1)$. Then

$$t_{(0,1)} \circ \overline{r}_M(x, y) = t_{(0,1)}(x, -y) = (x, -y + 1),$$

but

$$\overline{r}_M \circ t_{(0,1)}(x, y) = \overline{r}_M(x, y + 1) = (x, -y - 1).$$

so these two isometries don't agree.

(c) By a suitable isometry, we may assume that L is the x-axis. Then we have

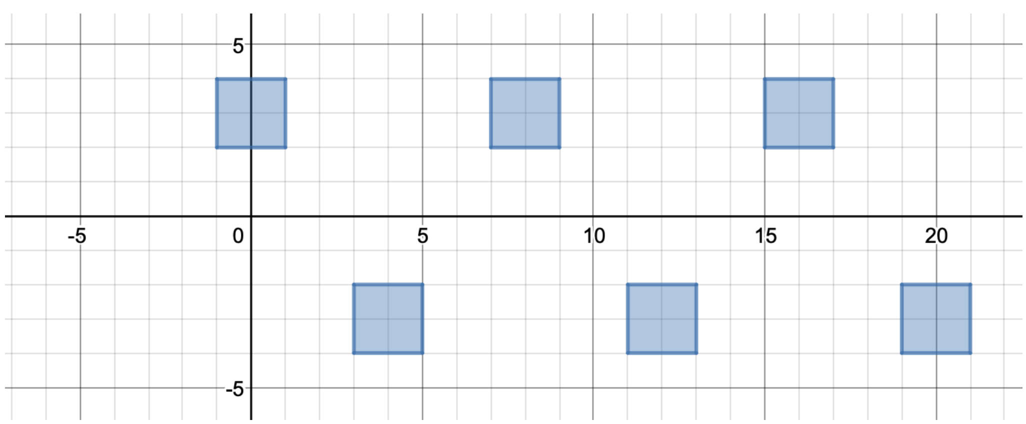
$$t_{(\alpha,\beta)} \circ \overline{r}(x, y) = (x + \alpha, -y + \beta),$$

and

$$\overline{r} \circ t_{(\alpha,\beta)}(x, y) = (x + \alpha, -y - \beta).$$

Since these two maps are the same, we conclude that $-y + \beta = -y - \beta$ for all $y \in \mathbb{R}$, so $\beta = 0$. Therefore, $(\alpha, \beta) = (\alpha, 0)$ is parallel to the x-axis.

(d) Shown below are the box B and its five iterated glides, each to the right of the previous.



(e) In the Theorem on pp. 12-13 of Stillwell, most cases are considered. You should make sure you understand them and can present them in your own argument. Here are the remaining cases.

Suppose L and M intersect in a point not on N. If M and N intersect, then they do so at a point not on L, so Case (ii) in the Theorem applies. Similarly, if L and N intersect in a point not on N, but N and M intersect, then Case (ii) applies again.

The only remaining case is when L is a transversal intersecting parallel lines M and N. Suppose L intersects M at P. Since $r_M r_L$ is a rotation, we may rotate lines M and L about P as a pair, making the replacement $r_M r_L = r_{M'} r_{L'}$, where M' and L' are any two lines intersecting at P, retaining the angle between M and L, but *distinct* from M and L. Now we no longer have parallel lines, and Case (ii) applies once again.

Remark: The key is always to find new lines $\widetilde{N}$, $\widetilde{M}$, and $\widetilde{L}$, with $\widetilde{M}$ and $\widetilde{N}$ perpendicular to $\widetilde{L}$, and such that $\overline{r}_N \overline{r}_M \overline{r}_L = \overline{r}_{\widetilde{N}} \overline{r}_{\widetilde{M}} \overline{r}_{\widetilde{L}}$. By the arguments in the theorem, this will always be a glide reflection.

Problem 4. (20 pts) The goal of this exercise is to complete the following table:

	Reflection $\overline{r}_L$	Translation $t_{(\alpha,\beta)}$	Rotation $R_{\theta,P}$	Glide reflection
Reflection $\overline{r}_M$				.
Translation $t_{(\gamma,\delta)}$				
Rotation $R_{\phi,Q}$				
Glide Reflection				

The table is completed as follows. At a given entry, we want to describe the type of isometry $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is obtained by composing an isometry $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the type indicated by its row with an isometry $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the type indicated by its column. There are a total of four types: reflections, translations, rotations and glide-reflections. There can be more than one type per entry.

In general, we will include reflections $\overline{r}_L$ within the set of glide reflections. Just for the purpose of this problem, *glide reflection* refers to a glide reflection which is not a reflection.

(a) Show that $\overline{r}_M \overline{r}_L$ is either a rotation or a translation. What is the geometric position between M and L if $\overline{r}_M \overline{r}_L$ is a translation ?

(b) Show that the composition of a glide reflection with a reflection is a rotation or a translation.

(c) Complete the table above.

(d) The order in which we compose isometries can matter. Show that a rotation and a reflection do not necessarily commute.

(e) Discuss whether glide-reflections commute with reflections, translations and rotations.

Solution. Let us ignore the identity function, so that our four categories are disjoint (remember that the identity is technically a translation *and* a rotation.

(a) It has been discussed (and proven in Stillwell) that $\overline{r}_M\overline{r}_L$ is a translation exactly when M and L are parallel, and a rotation otherwise.

(b) Consider a glide $\overline{r}_M\overline{r}_N\overline{r}_L$, where M and N are perpendicular to L. Composing on the left with $\overline{r}_M$ gives a rotation. From Problem 3(c), we know that $\overline{r}_M\overline{r}_N\overline{r}_L = \overline{r}_L\overline{r}_M\overline{r}_N$ is the same glide. Composing on the left with $\overline{r}_L$ gives a translation. These are all the possibilities for this composition. Similarly, composing $\overline{r}_M\overline{r}_N\overline{r}_L$ on the right by $\overline{r}_L$ gives a translation, and composing $\overline{r}_L\overline{r}_M\overline{r}_N$ on the right by $\overline{r}_N$ gives a rotation. So glides compose with reflections to give either kind of orientation-preserving isometry.

(c) We will use the classification from Problem 6(d) and (e) below, that parity must be respected. That is, rotations and translations can only combine among themselves, reflections and glides must combine to form rotations and translations, and mixing parity must result in a reflection or glide.

From part (a), we have the upper left entry. From Problem Set 1, Problem 4(c), we know the second diagonal entry. Problem 6(b) below shows that the composition of rotations may be a translation, and we also know that it may be a rotation.

Composing a translation after a reflection may be a glide, and the reverse order may be too. From the characterization of glides in Problem 3(b) and (c), we know there must be compositions of reflections and translations that are not glides, so they must be reflections.

From the proof of Problem 6(d), we know that the composition of a rotation and a translation, in either order, is a rotation.

Consider a rotation R followed by a reflection $\overline{r}_L$. By rotating the reflecting lines used to create the rotation R, we can assume that the second is parallel with L. Therefore, this composition is a reflection (the first line used in R) and a translation (the second and third). Depending on whether this first line is perpendicular to L or not, we can get a glide or a reflection. The reverse order shows the same result.

Finally, we consider the compositions with glides not covered in part (b). A glide can be thought of as ending or beginning with a translation, so the following result applies to composition on both sides. Compose a glide with a translation in its same direction. If that translation negates the translation of the glide, we get a reflection. Otherwise we get another glide. These are the only possibilities.

This also applies to glides on either side: draw the three lines of a glide. Two of the perpendicular lines may be thought of as occurring at the end of the glide or at the beginning. Either way, they represent a rotation which may be used to cancel them, leaving only one reflection. A more arbitrary rotation may be used that cancels only one reflection used in the translation in the glide, eventually giving three reflections in three lines which have no triple point, which we have shown to be a nontrivial glide.

By performing two glides which differ only in the length of translation (but are in the same direction), we obtain a translation. Finally, consider two glides in different directions, $t_1\overline{r}_L$ and $\overline{r}_{L'}t_2$. Composing them in the right way, we have

$$t_1 \circ (\overline{r}_L\overline{r}_{L'}) \circ t_2,$$

The composition of a translation, rotation, and translation, which we know to be a rotation from our chart. Therefore, glides can compose to rotations as well.

	Reflection $\overline{r}_L$	Translation $t_{(\alpha,\beta)}$	Rotation $R_{\theta,P}$	Glide reflection
Reflection $\overline{r}_M$	Rot, Tran	Refl, Glide	Refl, Glide	Rot, Tran
Translation $t_{(\gamma,\delta)}$	Refl, Glide	Tran	Rot	Refl, Glide
Rotation $R_{\phi,Q}$	Refl, Glide	Rot	Rot, Tran	Refl, Glide
Glide Reflection	Rot, Tran	Refl, Glide	Refl, Glide	Rot, Tran

Remark: We have shown that when composing isometries of opposite parity, in any order, it is always possible to obtain a reflection or a glide. The same is not true for the orientation-preserving isometries, as we see that some entries in our table include only rotations or translations, but not both. Interestingly, the only combinations in which we don't have "both" kinds of outcomes are the composition of translations with orientation-preserving isometries.

Also notice that also that our chart is symmetric. While isometries may not commute, neither direction of composition is "richer" than the other.

(d) Take the rotation $R_{\pi/2}$ and the standard reflection $\overline{r}$. Then

$$R_{\pi/2} \circ \overline{r}(x, y) = R_{\pi/2}(x, -y) = (y, x),$$

but

$$\overline{r} \circ R_{\pi/2}(x, y) = \overline{r}(-y, x) = (-y, -x).$$

so $R_{\pi/2} \circ \overline{r} \neq \overline{r} \circ R_{\pi/2}$.

(e) Notice (by calculating) that the glide $t_{(1,0)}\overline{r}$ does not commute with either the translation $t_{(0,1)}$, nor the reflection $\overline{r}_L$ in the y-axis, nor the rotation $R_{\pi/2}$.

Problem 5. (20 pts) Let $T \subseteq \mathbb{R}^2$ be the equilateral triangle centered at the origin.

(a) Show that there are exactly *six* isometries $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which verify $f(T) = T$. Let us call them $s_1, s_2, s_3, s_4, s_5, s_6$.

(b) Explain why the composition $s_i \circ s_j$, for any $1 \leq i, j \leq 6$, must be of the form s_k, for some $1 \leq k \leq 6$.

(c) Complete the table below according to this product rule: in entry ij in the table is s_k if $s_j \circ s_i = s_k$. In particular, explain why the set of isometries which preserve T form a group G_T.

	s_1	s_2	s_3	s_4	s_5	s_6
s_1						
s_2						
s_3						
s_4						
s_5						
s_6						

(d) Is the group G_T commutative, i.e. $s_i \circ s_j = s_j \circ s_i$, for all $1 \leq i, j \leq 6$?

(e) Consider the set $I = \{1, 2, 3\}$ with three elements. Show that there are exactly six bijections $F : I \longrightarrow I$. Let us call them $F_1, F_2, F_3, F_4, F_5, F_6$.

(f) Complete the following table, where in the (ij) entry we write the bijection F_k which corresponds to the composition $F_j \circ F_i$.

	F_1	F_2	F_3	F_4	F_5	F_6
F_1						
F_2						
F_3						
F_4						
F_5						
F_6						

(g) Show that there is a relabeling of $s_1, s_2, s_3, s_4, s_5, s_6$ into $F_1, F_2, F_3, F_4, F_5, F_6$ such that the two tables in Part 2.(c) and Part 2.(f) above coincide.

This proves that the group of isometries of the regular triangle is the same as the group of bijections of three elements.

(h) Let $S \subseteq \mathbb{R}^2$ the square with vertices $(1, 1), (1, -1), (-1, 1), (-1, -1) \in \mathbb{R}^2$. How many isometries $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are there such that $f(S) = S$?

Solution.

(a) Assume the result of part (e). Such an isometry must leave invariant the set of vertices of T, so there are only $3! = 6$ candidates for isometries, the different permutations of vertices. We must show that *all* of these permutations are actually realized by isometries. Let's fix T to have vertices on the unit circle at angles 0, $2\pi/3$, and $4\pi/3$ (the cube roots of unity in the complex plane). Let L_1 be the x-axis, L_2 be the line through the origin at angle $2\pi/3$, and L_3 be the line through the origin at $4\pi/3$. Notice that the six isometries

$$s_1 = \text{Id}, \;\; s_2 = R_{2\pi/3}, \;\; s_3 = R_{4\pi/3}, \;\; s_4 = L_1, \;\; s_5 = L_2, \quad \text{and} \quad s_6 = L_3$$

each realize a different permutation of the vertices of T.

(b) You can verify this using theorems about rotations and reflections, or by calculating all of the formulas, but here's a cute way: the composition of two permutations (meaning, you mix up the numbers 1, 2, and 3, ordered, and then do it *again*) is of course another permutation. Since isometries are uniquely determined by where they send three points, and since every permutation of the vertices is in our set $\{s_1, s_2, \ldots, s_6\}$, any composition $s_i \circ s_j$ will correspond to a permutation realized by some other s_k in the set.

(c) Your table may look different from mine, if you labeled your isometries s_i differently from me. But the first filled column and row should be the same (why?).

	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_2	s_3	s_4	s_5	s_6
s_2	s_2	s_3	s_1	s_5	s_6	s_4
s_3	s_3	s_1	s_2	s_6	s_4	s_5
s_4	s_4	s_6	s_5	s_1	s_3	s_2
s_5	s_5	s_4	s_6	s_2	s_1	s_3
s_6	s_6	s_5	s_4	s_3	s_2	s_1

Let's check that we have a group. As always, we have the binary operation of function composition. In part (b) we checked the closure property (that the binary operation actually maps into the set we care about). Function composition is always associative. The isometry s_1 works as a valid identity element because $\mathrm{Id} \circ f = f \circ \mathrm{Id} = f$ for any function. Finally, every element has a valid inverse. The reflections are their own inverses, and the two nontrivial rotations are inverses of each other.

(d) No, G_T is a *nonabelian* group. In particular,

$$s_2 \circ s_4 = s_6 \neq s_5 = s_4 \circ s_5.$$

Remark: What we have just constructed is called the multiplication table for our group. Every (finite) group has one, and you can always tell whether a group is abelian or not by checking if the table is symmetric (in the matrix sense). It is important that G_T here is not abelian. Play around with the isometries to see why they don't commute.

(e) In any set of n elements, such a function is determined by its n images. The image of 1 has n choices (any of the numbers $1, \ldots, n$). The function is bijective, so $F(2) \neq F(1)$, thus we have $n-1$ choices for this image. Next, $F(3)$ will have $n-2$ possibilities. Continuing, we finish by finding 2 choices for $F(n-1)$ and only one choice left for $F(n)$. Therefore, we have

$$n(n-1)(n-2)\ldots(2)(1) = n!$$

choices, and these are the possible number of bijections. In our case $n = 3$, so there are $3! = 3 \cdot 2 \cdot 1 = 6$ bijections. These are called permutations. Let's give them labels. F_1 is the identity function. F_2 is the function that sends $1 \mapsto 2$, $2 \mapsto 3$, and $3 \mapsto 1$. F_3 is the function that sends $1 \mapsto 3$, $3 \mapsto 2$, and $2 \mapsto 1$. F_4 is the function that fixes only 1 (so it flips 2 with 3), F_5 fixes only 2, and F_6

fixes only 3.

(f) I have chosen the labels so that the function function F_i to corresponds exactly to how the isometry s_i permutes the vertices, so the tables are identical.

	F_1	F_2	F_3	F_4	F_5	F_6
F_1	F_1	F_2	F_3	F_4	F_5	F_6
F_2	F_2	F_3	F_1	F_5	F_6	F_4
F_3	F_3	F_1	F_2	F_6	F_4	F_5
F_4	F_4	F_6	F_5	F_1	F_3	F_2
F_5	F_5	F_4	F_6	F_2	F_1	F_3
F_6	F_6	F_5	F_4	F_3	F_2	F_1

(g) Due to the choices I made, the labeling is the boring one: $s_i \mapsto F_i$. In general, you want to relabel your s_i as the F_i that corresponds to the way that s_i permutes the vertices, after you've given your vertices the names 1, 2, and 3. Here I am thinking of the vertices T as labeling the vertex on the x axis as 1, and then continuing to increase the vertex number by 1 as I move counterclockwise.

Remark: The group of bijections of n elements is called *the group of permutations on n letters* or S_n. The group of symmetries of the regular n-gon is called the *dihedral group*, or D_n. What we have shown is that the group S_3 is *isomorphic* to D_3, meaning, informally, that there is a relabeling of the elements of one into the elements of the other such that the multiplication tables are the same.

(h) There are 8, the four rotations (including the trivial one) and four reflections found in Problem Set 1, Problem 6, parts (b) and (d). Here is a quick way to see that, given the proofs in the solutions to that problem. Any isometry preserving the square must fix the origin, which limits us to rotations about the origin and reflections in lines through origin. But any other rotation or reflection through the origin, other than the 8 considered here, has been proved to not preserve the square.

Alternatively, you could go the permutation rout. There are $4! = 24$ possible permutations, but not all are possible, because many of them would take opposite corners to adjacent corners, shortening their distance, so they cannot be isometries. This didn't happen with the triangle because *all* points on T were the exact same distance from all other points.

Remark: Consider thinking about the number of elements of a general dihedral group for a polygon with n sides. Can you find a pattern? Does it continue to be true that S_n is isomorphic to D_n, past $n = 3$?

Problem 6. (20 pts) For each of the ten sentences below, justify whether they are **true** or **false**. If true, you must provide a proof, if false you must provide a counter-example.

(a) Planar Isometries preserve angles. That is, let $O, P, Q \in \mathbb{R}^2$ be points and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ an isometry. Then the angle between the vectors $\vec{OP}$ and $\vec{OQ}$ equals the angle between the vectors $f(O)\vec{f}(P)$ and $f(O)\vec{f}(Q)$.

(b) The set of rotations $R_{\theta,P} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ form a group inside the isometry group of the Euclidean plane.

(c) The set of translations $t_{(\alpha,\beta)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ form a group inside the isometry group of the Euclidean plane.

(d) Let us call an isometry *orientation-preserving* if it is the product of two reflections. The set of orientation-preserving isometries is a group.

(e) Let us call an isometry *orientation-reversing* if it is the product of one or three reflections. The set of orientation-reversing isometries is a group.

(f) Suppose that $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are isometries and $A, B, C \in \mathbb{R}^2$ are three points. If $f(A) = g(A)$, $f(B) = g(B)$ and $f(C) = g(C)$, then $f = g$.

(g) Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be isometries that fix *all* points of the same line $L \subseteq \mathbb{R}^2$. Then it must be that $f = g$.

(h) Let T be the triangle in Problem 5, and $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be isometries such that the vertices of the triangle $f(T)$ coincide with the vertices of the triangle $g(T)$. Then $f = g$.

(i) Let T be the triangle in Problem 5, and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry such that $f(P) = P$ for $P \in T$. Then $f(Q) = Q$ for all $Q \in \mathbb{R}^2$.

(j) Let $A, B, C, D \in \mathbb{R}^2$ be four points. There always exists a point $P \in \mathbb{R}^2$ such that $d(P, A) = d(P, B) = d(P, C) = d(P, D)$.

Solution.

(a) True. We have kind of been using this implicitly whenever we say "by a suitable isometry, assume..." or "there is a change of coordinates such that...". If our points are not collinear, then they form a triangle. The side lengths of this triangle are preserved by our isometry f, so by the Side-Side-Side theorem of grade school geometry, the image of the triangle is congruent to the original triangle, and therefore the angles are preserved.

If our points are on a line, then they form the angle 0 or π, depending on which one is in the middle. Isometries map lines to lines, so the image of these points also forms an angle of 0 or π. The angle can't switch from 0 to π (or vice versa), because to do so would require that f reorder points on a line, which would not preserve distance.

(b) False. Problem 4 (a) and (b) of Problem Set 1 show that this is true for a *fixed* value of P, just varying θ, but the result fails if we are allowed to compose rotations about different points. Take distinct parallel lines M and N, and a third line transverse to these, L. Then $\overline{r}_L\overline{r}_M$ and $\overline{r}_N\overline{r}_L$ are both rotations, but

$$(\overline{r}_N\overline{r}_L) \circ (\overline{r}_L\overline{r}_M) = \overline{r}_N \circ (\overline{r}_L\overline{r}_L) \circ \overline{r}_M = \overline{r}_N \circ \mathrm{Id} \circ \overline{r}_M = \overline{r}_N\overline{r}_M$$

is a nontrivial translation, which we have argued cannot be a rotation.

(c) True. This was the content of Problem Set 1, Problem 4, parts (c) and (d), along with the observation that the identity is a translation (by the zero vector).

(d) True. We denote this subgroup of $\mathrm{Iso}(\mathbb{R}^2)$ by $\mathrm{Iso}^+(\mathbb{R}^2)$. We need to check identity, and inverses, and closure. The identity is the product $\overline{r}_L\overline{r}_L$ for any line L, so it preserves orientation. Any product of two reflections $\overline{r}_L\overline{r}_M$ has the inverse

$$(\overline{r}_L\overline{r}_M)^{-1} = \overline{r}_M^{-1}\overline{r}_L^{-1} = \overline{r}_M\overline{r}_L,$$

which preserves orientation.

Finally, to show closure, consider the orientation preserving isometries $\overline{r}_{L_1}\overline{r}_{L_2}$ and $\overline{r}_{L_3}\overline{r}_{L_4}$ through four arbitrary lines. If either is the identity, then their composition clearly preserves orientation. Assume then that neither is the identity, meaning $L_1 \neq L_2$ and $L_3 \neq L_4$. We wish to show that

$$\overline{r}_{L_1}\overline{r}_{L_2} \circ \overline{r}_{L_3}\overline{r}_{L_4}$$

is either a rotation or a translation (because both of these preserve orientation). If $\overline{r}_{L_1}\overline{r}_{L_2}$ and $\overline{r}_{L_1}\overline{r}_{L_2}$ are both translations, then their composition is a translation by part (c). If they are both rotations, then we are in the situation of the counterexample given in the solution to part (b), so their composition is a translation by the calculation given there.

Suppose $\overline{r}_{L_1}\overline{r}_{L_2}$ is a rotation and $\overline{r}_{L_3}\overline{r}_{L_4}$ is a translation. Recall that we can factor translations and rotations in many ways. Therefore, by refactoring, we can assume that L_1 and L_2 are rotated so that L_2 is parallel to L_3 and L_4 (notice that L_3 and L_4 are parallel by assumption). Then by refactoring, translate L_3 and L_4 so that L_3 coincides with L_2. Similarly, if $\overline{r}_{L_1}\overline{r}_{L_2}$ is a translation and $\overline{r}_{L_3}\overline{r}_{L_4}$ is a rotation, then we can translate the first pair of lines and rotate the second pair to make $L_3 = L_2$. In either case,

$$\begin{aligned}
\overline{r}_{L_1}\overline{r}_{L_2} \circ \overline{r}_{L_3}\overline{r}_{L_4} &= \overline{r}_{L_1} \circ (\overline{r}_{L_2}\overline{r}_{L_3}) \circ \overline{r}_{L_4} \\
&= \overline{r}_{L_1} \circ (\overline{r}_{L_2}\overline{r}_{L_2}) \circ \overline{r}_{L_4} \\
&= \overline{r}_{L_1} \circ \mathrm{Id} \circ \overline{r}_{L_4} \\
&= \overline{r}_{L_1}\overline{r}_{L_4},
\end{aligned}$$

which is a rotation, because L_1 cannot be parallel to L_4 (because both constructions ended up making the first three lines parallel, or the last three lines parallel, so if L_1 and L_4 are parallel, then all four lines are parallel, and then we never had a rotation).

(e) False. In particular, this set does not contain the identity.

Alternatively, the product of two orientation-reversing isometries will *always* be an *orientation-preserving* isometry. In particular, the product of any two reflections is *two* reflections, which can never be written as the product of one or three reflections. Therefore, the binary operation of function composition is not *closed* on the set of orientation-reversing isometries.

(f) False. This looks like a Theorem that we have discussed, except that it is missing the crucial assumption that A, B, and C are not *collinear*. This suggests that we pick collinear points as a counterexample. Indeed, suppose these points lie on a line L. Then take $f = \text{Id}$ and $g = \overline{r}_L$. Clearly $f \neq g$, but all three points are fixed by each, meaning $f(A) = A = g(A)$, $f(B) = B = g(B)$, and $f(C) = C = g(C)$.

(g) False. Again take $f = \text{Id}$ and $g = \overline{r}_L$.

(h) False. In Problem 5 above, we constructed six *distinct* such isometries. Each one created images of the triangle with identical *sets* of vertices, but no two were the same isometry.

(i) True. Take $g = \text{Id}$ in part (h). Then the vertices of $f(T)$ coincide with the vertices of $\text{Id}(T) = T$, so $f = \text{Id}$, and therefore fixes all points in the plane.

(j) False. We will prove a stronger result, that you can't always find such a P even if you only require equal distances from *three* points. Let A, B, and C be distinct points on a common line L. The points equidistant from A and B form a line M. The reflection $\overline{r}_L$ exchanges A and B, so L is perpendicular to M. Repeat this for the pair of points B and C, obtaining the line N of points equidistant from B and D, which is again perpendicular to L. Since $C \neq A$, we have that $N \neq M$ are two nonintersecting lines. Therefore, there is no point P equidistant from A, B, and C. Meaning it is impossible to have

$$d(P, A) = d(P, B) = d(P, C).$$

Problem 7. (20 pts) In this problem we will explore *wall-paper* tilings of the Euclidean plane. There is a classification of *all* possible regular tiling of the plane, resulting in 17 isometry subgroup of the group of isometries of $\mathbb{R}^2$. In this exercise we will study some of them. Each of the patterns displayed in the Figures should be understood as extending infinitely all over the Euclidean Plane $\mathbb{R}^2$.

(a) Explore whether the pattern in Figure 1 admits any translational symmetries, i.e. it is invariant under translations in the Euclidean plane. Is it invariant under any reflection ?

Figure 1. The pattern in the Patio de los Arrayanes in *La Alhambra*, Spain.

(b) Explore whether the pattern in Figure 2 admits any rotational symmetries, i.e. whether it is invariant under certain rotations of the Euclidean plane. Are there any reflection which preserve this pattern ?

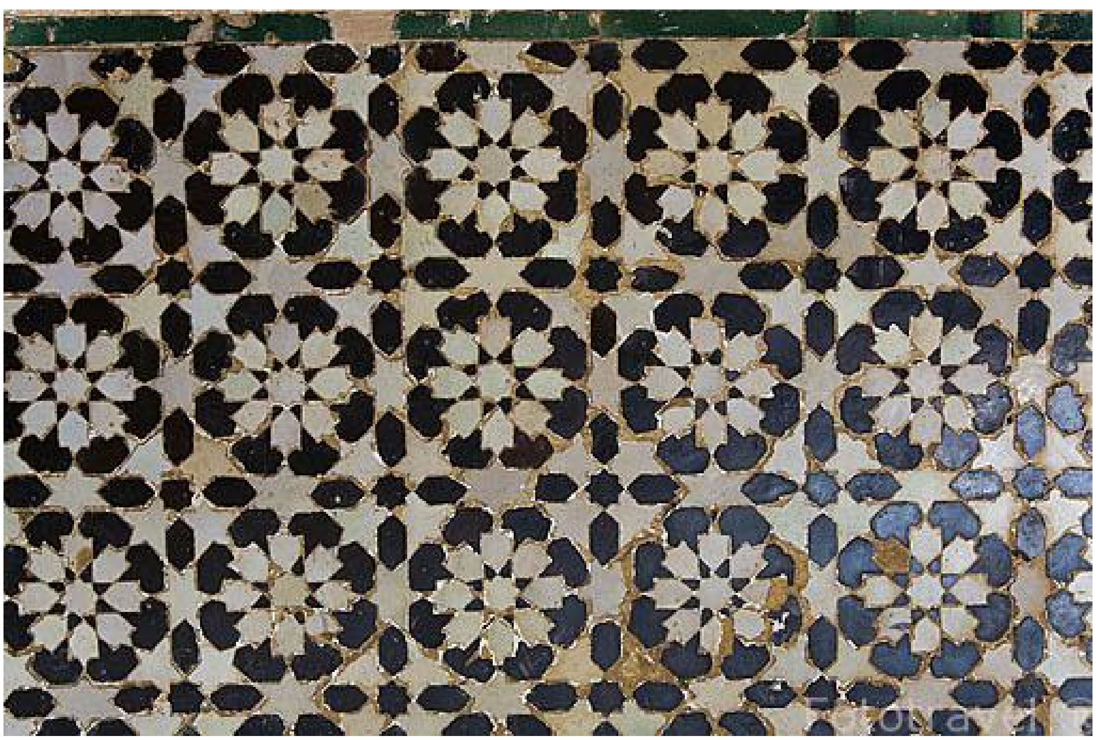


Figure 2. The pattern in La Torre de las Infantas in *La Alhambra*, Spain.

(c) Find all translations which leave the pattern in Figure 3 invariant.

FIGURE 3. A tiling of the plane made of fish and birds.

(d) Compare the isometries which preserve the brick pattern in Figure 4 with the isometries that preserve the tile pattern in Figure 5.

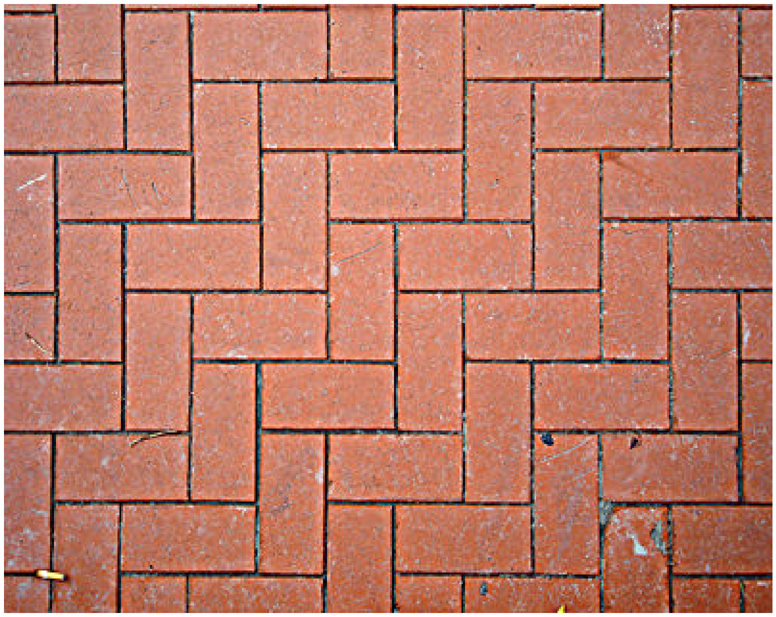


FIGURE 4. First Brick Tiling.

FIGURE 5. Second Brick Tiling.

(e) (Bonus - Extra Credit) Explore the isometries of the patterns in Figures 6, 7, 8 and 9.

Figure 6. M.C. Escher Birds.

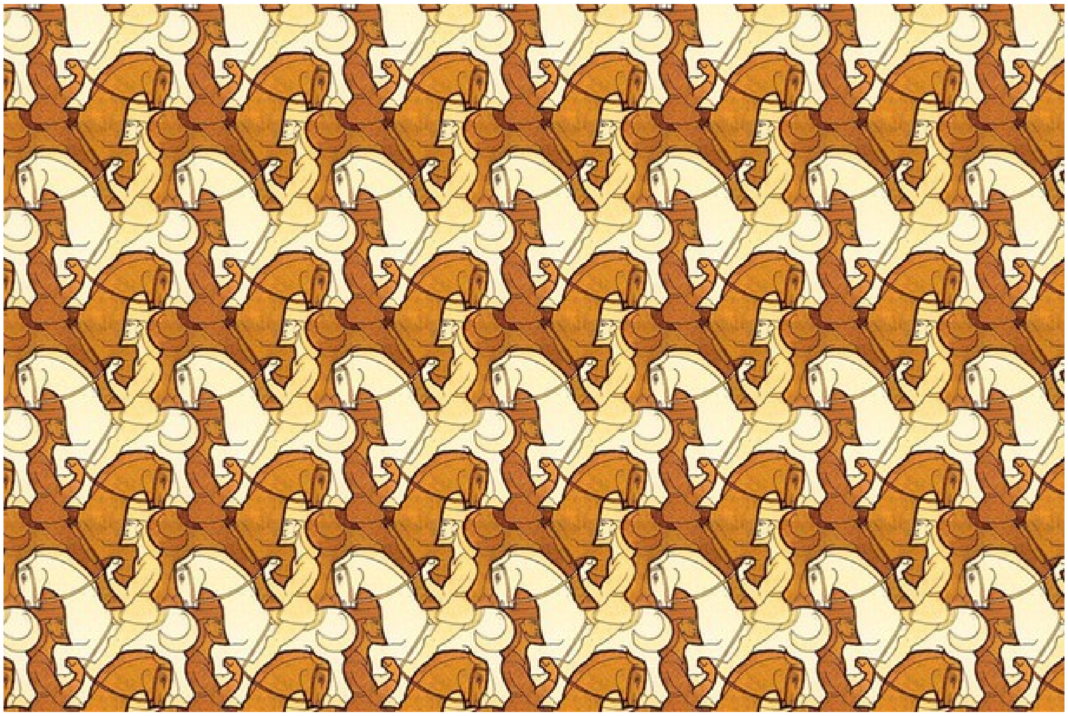


FIGURE 7. General in Horse Tiling.

FIGURE 8. M.C. Escher's fish.

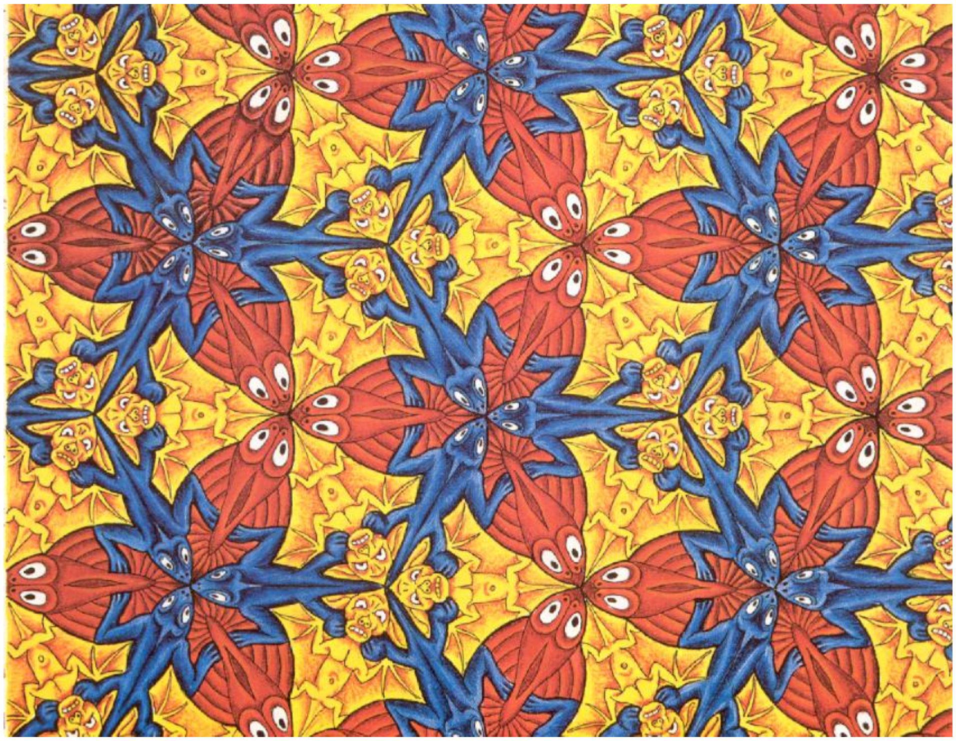


FIGURE 9. M.C. Escher's Artwork.

Solution.

Take a look at the Wikipedia page on wallpaper groups to see the classification of these 17 subgroups of $\mathrm{Iso}(\mathbb{R}^2)$. See if you can decide which group each of these tilings correspond to.

(a) Fig. 1 admits two independent directions of translational symmetry (focusing on a six-pointed star, one may shift it to the next one to the right, or the next one to the upper-right). Any reflection would reverse the orientation of the pin-wheels (which point counter-clockwise now), so Fig. 1 admits no reflection symmetries.

(b) There is some local symmetry in rotation by $\pi/4$ in Fig. 2, but this isn't good enough to preserve the whole tiling, whose main breakdown is into squares. We can however conclude that this tiling has symmetry by rotation $\pi/2$. Additionally, there is reflection symmetry through the usual four lines which preserve the square (notice that these generate the rotations by $\pi/2$ anyway).

(c) Similar to Fig. 1, we have two independent directions of translational symmetry in Fig. 3. All translations are generated by combinations of the translation T_1 which sends a fish to the next one above and to the left, and T_2 which sends a fish to the next one above and to the right.

(d) These are similar. Say the bricks have dimensions 2 units by 1 unit. The translational symmetries of Fig. 4 are generated by $t_{(-1,1)}$ and $t_{(4,0)}$, while those of Fig. 5 are generated by $t_{(1,1)}$ and $t_{(4,0)}$. In fact, these figures differ only by a single reflection, (reflection in either the x-axis or the y-axis will work). Therefore, neither has rotational symmetry, because that would change the direction that the bricks "point". Similarly, neither have reflection symmetry, because such a symmetry would have to be parallel to the direction of pointing, but even still would not preserve the pattern of lines made by the bricks.

(e) Much can be said about each. Note that *all* wallpaper groups have two independent directions of translation. Particularly interesting is that Fig. 8 does not have rotational symmetry by $\pi/3$ (it would change the orientation of the fish fins), but does have rotational symmetry by $2\pi/3$. The same is true of Fig. 9.