SOLUTIONS TO PROBLEM SET 3

MAT 141

Abstract. These are the solutions to Problem Set 3 for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. The problems were posted online on Friday Jan 24 and due Friday Jan 31 at 10:00am.

Problem 1. Let $\Gamma \subseteq \text{Iso}(R^2)$ be a subgroup of the isometry group of the Euclidean Plane $\mathbb{R}^2$. Consider the set of Γ-orbits $\mathbb{R}^2/\Gamma$ and define the distance $\mathbb{R}^2/\Gamma$ as follows:

$$d(\Gamma(P), \Gamma(Q)) := \min_{P', Q' \in \mathbb{R}^2} \{d(P', Q') : P' \in \Gamma(P), Q' \in \Gamma(Q)\},$$

where $P, Q \in \mathbb{R}^2$. Show that we have the equality

$$d(\Gamma(P), \Gamma(Q)) = \min_{Q' \in \mathbb{R}^2} \{d(P, Q') : Q' \in \Gamma(Q)\}.$$

Solution. It suffices to show that the two *sets* are actually equal. That is, we need to show that, given $d(P', Q')$ for $P' \in \Gamma(P)$ and $Q' \in \Gamma(Q)$, we can find some other $Q'' \in \Gamma(Q)$ such that $d(P, Q'') = d(P', Q')$. Well, $P' = g(P)$ for some group element $g \in \Gamma$, so take $Q'' = g^{-1}(Q')$. Note that $Q'' \in \Gamma(Q') = \Gamma(Q)$. Then, because all group elements are isometries (and invertible), we have

$$d(P, Q'') = d(g^{-1}(P'), g^{-1}(Q')) = d(P', Q').$$

Problem 2. Let $S \subseteq \text{Iso}(\mathbb{R}^2)$ be a subset of the isometry group, and let $\langle S \rangle \subseteq \text{Iso}(\mathbb{R}^2)$ denote the group generated by elements in S.

(a) If $S = \{t_{(1,0)}\}$ show that

$$\langle t_{(1,0)} \rangle = \{t_{(n,0)} : n \in \mathbb{Z}\}.$$

(b) If $S = \{t_{(1,0)}, t_{(0,1)}\}$ show that

$$\langle S \rangle = \{t_{(n,m)} : (n, m) \in \mathbb{Z}^2\}.$$

(c) If $S = \{t_{(1,0)}, t_{(0,1)}\}$, $\Gamma = \langle S \rangle$, and $P = (0, 0)$ show that

$$\Gamma(P) = \{(n, m) \in \mathbb{Z}^2\}.$$

(d) If $S = \{t_{(1,0)} \circ \overline{r}\}$, $\Gamma = \langle S \rangle$, and $P = (0, 0)$ show that

$$\Gamma(P) = \{(n, 0) \in \mathbb{Z}^2\}.$$

(e) If $S = \{t_{(1,0)} \circ \overline{r}\}$, $\Gamma = \langle S \rangle$, and $P = (0, 1)$ show that

$$\Gamma(P) = \{(n, (-1)^n) \in \mathbb{Z}^2\}.$$

Solution. Recall that the subgroup generated by a subset S is the set of isometries obtained by composing the elements of S (and their inverses) in all possible ways, over and over.

(a) The subgroup generated by a single element is just all (positive, negative, and zero) powers of that element, so

$$\langle t_{(1,0)}\rangle = \{t_{(1,0)}^n : n \in \mathbb{Z}\} = \{t_{(n,0)} : n \in \mathbb{Z}\}.$$

(b) In general, the subgroup generated by more than one element is the set of *all* possible strings of powers of those elements, written in any order. But these two elements commute, so all of their powers will commute with each other too. Therefore, we can sort any string such that all powers of $t_{(1,0)}$ appear on the left, and all powers of $t_{(0,1)}$ appear on the right. Therefore,

$$\langle t_{(1,0)}, t_{(0,1)}\rangle = \{t_{(1,0)}^n t_{(0,1)}^m : n \in \mathbb{Z}, m \in \mathbb{Z}\} = \{t_{(n,m)} : (n,m) \in \mathbb{Z}^2\}.$$

(c) Recall that the Γ-orbit of a point P is defined to be $\Gamma(P) = \{g(P) \mid g \in \Gamma\}$. Using part (b), we have

$$\Gamma(P) = \{t_{(n,m)}(0,0) : (n,m) \in \mathbb{Z}^2\} = \{(n,m) : (n,m) \in \mathbb{Z}^2\} = \{(n,m) \in \mathbb{Z}^2\}.$$

(d) First we calculate $\langle S\rangle$. The set S has only one element, so we only need to consider all powers of this glide. Notice that

$$(t_{(1,0)}\overline{r})^2 = (t_{(0,1)}\overline{r})(t_{(1,0)}\overline{r}) = (t_{(1,0)}\overline{r})(\overline{r}(t_{(1,0)}) = t_{(2,0)},$$

and subsequently we have

$$(t_{(1,0)}\overline{r})^{2n} = t_{(2n,0)}$$

for all positive n. You can check in the same way that this formula actually holds for all $n \in \mathbb{Z}$. For odd powers, we multiply this formula by one more glide, giving

$$(t_{(1,0)}\overline{r})^{2n+1} = t_{(2n+1,0)}\overline{r}$$

for all $n \in \mathbb{Z}$. The even powers act on $(0,0)$ to give $t_{(2n,0)}(0,0) = (2n,0)$, and the odds act to give $t_{(2n+1,0)}\overline{r}(0,0) = (2n+1,0)$. Ranging over n gives the result desired.

(e) Apply the same technique as in part (d). The even powers give

$$t_{(2n,0)}(0,1) = (2n,1) = (2n,(-1)^{2n}),$$

and the odds give

$$t_{(2n+1,0)}\overline{r}(0,1) = (2n+1,-1) = (2n+1,(-1)^{2n+1}).$$

The result is then obtained by ranging over n.

Problem 3. (20 pts) (**The Cylinder**) Let $C = \mathbb{R}^2/\Gamma$ be the Euclidean cylinder, where $\Gamma = \langle t_{(0,1)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the translation $t_{(0,1)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. We shall use coordinates $(x, y) \in C$, induced by the coordinates $(x, y) \in \mathbb{R}^2$, where in the cylinder C we identify $(x, y) \sim (x + n, y)$ for any $n \in \mathbb{Z}$.

Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the cylinder C is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) For each of the following four pairs of points $P, Q \in C$, compute the distance $d_C(P, Q)$ between these points in the cylinder.

$$P = (0.5, 0.5), Q = (0.5, 0.7), \qquad P = (0, 0), Q = (0, 0.3)$$
$$P = (0.1, 0.2), Q = (0.9, 0.3), \qquad P = (0.1, 0.1), Q = (0.9, 0.8).$$

(b) What is the distance between $(1/3, 1/2)$ and $(7/3, -11/2)$? And the distance between $(0, 0)$ and $(3, 4)$?

(c) Give example of three points $P \in \mathbb{R}^2$ such that $\pi(P) = (0, 0.5)$.

(d) Find infinitely points $P \in \mathbb{R}^2$ such that $\pi(P) = (0.2, 4) \in C$.

(e) (Exercise 2.2.1) Which of the following properties of euclidean lines hold for lines on the cylinder ?

(i) There is a line through any two points.

(ii) There is a unique line through any two points.

(iii) Two lines meet in at most one point.

(iv) There are lines which do not meet.

(v) A line has infinite length.

(vi) A line gives the shortest distance between two points.

(vii) A line does not cross itself.

Solution.

(a) For each of the following four pairs of points $P, Q \in C$, compute the distance $d_C(P, Q)$ between these points in the cylinder.

$$P = (0.5, 0.5), Q = (0.5, 0.7), \qquad P = (0, 0), Q = (0, 0.3)$$
$$P = (0.1, 0.2), Q = (0.9, 0.3), \qquad P = (0.1, 0.1), Q = (0.9, 0.8).$$

After playing around with these enough, you'll see that the distance can be computed similar to the usual $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, but where the term

$(x_2 - x_1)$ is replaced by its difference from its closest integer. In particular, the horizontal distance in C between $x = 0.1$ and $x = 0.9$ is 0.2. We find

$$\begin{aligned} d_C((0.5,0.5),(0.5,0.7)) &= \sqrt{0^2 + (0.2)^2} = 0.2 \\ d_C((0,0),(0,0.3)) &= \sqrt{0^2 + (0.3)^2} = 0.3 \\ d_C((0.1,0.2),(0.9,0.3)) &= \sqrt{(0.2)^2 + (0.1)^2} = \sqrt{0.05} \approx 0.224 \\ d_C((0.1,0.1),(0.9,0.8)) &= \sqrt{(0.2)^2 + (0.7)^2} = \sqrt{0.53} \approx 0.728. \end{aligned}$$

Remark: Notice that $d_C(P,Q) \leq d(P,Q)$ by definition (this is true for any quotient), with equality holding if and only if the x-coordinates of P and Q differ by at most $\frac{1}{2}$. (Notice that sometimes we just write P when we really mean $\Gamma(P)$. This abuse of notation is usually unambiguous given the context, and it is justified by our thinking of the map π, which takes P to $\Gamma(P)$, as "natural".)

(b) Both of these pairs differ in the horizontal direction by an integer amount, so their horizontal distances become zero in C. We have

$$\begin{aligned} d_C((1/3,1/2),(7/3,-11/2)) &= \sqrt{0^2 + 6^2} = 6 \\ d_C((0,0),(3,4)) &= \sqrt{0^2 + 4^2} = 4. \end{aligned}$$

(c) Understand that when we write $\pi(P) = P'$, we mean $P' \in \pi(P)$. Though $\pi(P) = \Gamma(P)$ is an equivalence *class*, we think of it as a point which can be represented by any of its members. Well,

$$\pi(0.5) = \{(n, 0.5) : n \in \mathbb{Z}\},$$

so we can pick any three points in here, say $(-1, 0.5)$, $(0, 0.5)$, and $(1, 0.5)$. This means that all three of these points in $\mathbb{R}^2$ become the *same* point in C.

(d) As before, the entire class $\pi(0.2, 4) = \{(n + 0.2, 4) : n \in \mathbb{Z}\}$ will do the trick.

(e) (i) True. For any two points in C, take any of their representatives in $\mathbb{R}^2$. The line L through these representatives descends to a line $\pi(L)$ in C by definition.

(ii) False. The choice of representatives mentioned in (i) above may not be unique. For example. take the points $P = \pi(0,0)$ and $Q = \pi(1,1)$ in the cylinder. We can represent these points in $\mathbb{R}^2$ by $(0,0)$ and $(1,1)$ if we like, in which case we will get a line L with positive slope. Notice that the line $\pi(L)$ in C spirals up as we move counterclockwise around the cylinder, and it includes the point $\pi(0.9, 0.9)$.

However, we could have also taken the representatives $(0,0)$ and $(-1,1)$. This gives us a line L' with negative slope. Then $\pi(L')$ is a line in the cylinder which spirals down as we move counterclockwise, and it does not contain $\pi(0.9, 0.9)$. If it did, then $(0.9, 0.9)$ would have to be equivalent (under the quotient) to some point on L'. But all of the points on L' are of the form $(-y, y)$, while no points in $\pi(0.9, 0.9)$ are of this form. We conclude that $\pi(L)$ and $\pi(L')$ are two *distinct* lines that both pass through P and Q.

Remark: In our example, we could have used other representatives of Q to obtain many other distinct lines. Notice that these lines retain their slopes, so lines with different slopes in $\mathbb{R}^2$ are still distinguished in C. We could have achieved lines with all possible slopes of the form $1/n$ for $n \in \mathbb{Z}$. So, in fact, there are *infinitely* many lines between these two points.

In general, points on the cylinder will have a unique line through them if and only if they have the same y-coordinates or the same x-coordinates. The former type create straight vertical lines, and the latter type create horizontal "circular" lines. All others create spirals. Intuitively, there are many ways to spiral around and connect two points that are related diagonally. You could do it fast (loosely winding) or slow (tightly wound).

(iii) False. The easiest way to see this is to construct two lines in $\mathbb{R}^2$, translate them horizontally by integer units to see all of their equivalent lines in C, and then note that these copies cross in points that are *not* identified in C. The lines L and L' from our solution to (ii) will do, because they both pass through $P = \pi(0, 0)$ and $Q = \pi(1, 1)$, which are different points in C. In fact, these lines meet in infinitely many points, all $\pi(n, n)$ for $n \in \mathbb{Z}$.

(iv) True. Two lines which never meet in $\mathbb{R}^2$ also never meet in C. This includes pairs of vertical lines, pairs of circular lines, and pairs of spirals of the same slope. Therefore, parallel-ness is preserved in this quotient, so t

(v) False. The image under π of any horizontal line L in $\mathbb{R}^2$ is a "circle" in C, and therefore has length equal to the circumference of our cylinder, which is 1.

Remark: Strictly speaking, we have not defined the length of a curve in C yet, but it will be something similar to the integral used in vector calculus for curves in Euclidean space. For now, we could say that L does not have infinite length because the set of distances between points on L is bounded above by 1 (whereas, a line with infinite length should have no such bound at all). Note that a line in C has finite length if and only if it is of this form.

(vi) True. Again, this requires a definition of the length of a line segment, but we can see the idea intuitively. Given points P and Q in C, look at the set of their equivalent points in $\mathbb{R}^2$. Choose some pair of these representatives, some $P' \in \pi(P)$ and $Q' \in \pi(Q)$, such that their distance achieves the minimum used in the definition of $d_C(P, Q)$. The line between P' and Q' in $\mathbb{R}^2$ descends in C to a line L between P and Q which achieves this shortest distance.

Note that this line may not be unique, as two spirals of opposite slope will achieve the same distance between points. Note also that not all lines through two points in C may achieve the necessary minimum distance. Think of tight spirals and loose spirals; the loose ones give shorter paths.

Remark: This is the unifying idea behind definitions of new geometries. We always want the distance between two points to be achieved by a line, so we *define* our notions of "distance" and "line" to make it so. Lines that achieve these distances are called *geodesics* in general. In Euclidean space, geodesics are the only lines that connect two points, but we have seen that this is not true in C.

(vii) True. Our lines are verticals, spirals, and circles. Certainly the infinite ones do not cross themselves. In thinking about circles, it kind of matters how we define "cross". The notion of crossing only makes sense if we discuss parameterizations: a curve L is called *simple* if L is the image of a continuous one-to-one parametrization, and we say that L crosses itself if it is not simple. Certainly a circle is simple, though you need a different parametrization than the one given by the quotient of a one-to-one parametrization for a horizontal line in $\mathbb{R}^2$, as this would not be one-to-one on in C. You could take just a segment of this parametrization however.

On the other hand, given any circle S in C, any representative line L for S in $\mathbb{R}^2$ necessarily contains infinitely many points which become the same point in S, and this is not true for the other types of lines in C. In this sense, you might say there is degeneracy. Still, we should feel comfortable saying that circles do not cross themselves!

Problem 4. (20 pts) (**The Möbius Band**) Let $M = \mathbb{R}^2/\Gamma$ be the Euclidean twisted cylinder, where $\Gamma = \langle t_{(1,0)} \circ \overline{r} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the glide reflection $t_{(1,0)} \circ \overline{r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. We shall use coordinates $(x, y) \in M$, induced by the coordinates $(x, y) \in \mathbb{R}^2$, where in the twisted cylinder M we identify $(x, y) \sim (x + n, (-1)^n y)$, for any value $n \in \mathbb{Z}$.

Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the twisted cylinder M is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) Let $P = (0.1, 0), Q = (0.9, 0.1), R = (0.1, 0.9), S = (-0.1, -0.9), T = (0.9, 0.9)$ be points in the twisted cylinder M. Find their pairwise distances.
(There are ten of them.)

(b) Consider the lines $L = \{x = 0.5\} \subseteq \mathbb{R}^2$ and $K = \{x = y\} \subseteq \mathbb{R}^2$. Find the intersection points of $\pi(L)$ and $\pi(K)$.

(c) Let $N = \{(x, y) \in \mathbb{R}^2 : x = -y\} \subseteq \mathbb{R}^2$. Find the number of intersection points of $\pi(K)$ and $\pi(N)$.

(d) Find two lines in the twisted cylinder M which do *not* intersect.

(e) Is it possible for two lines L_1, L_2 in the twisted cylinder M to have exactly one intersection point, i.e. $|L_1 \cap L_2| = 1$?

(f) Show that there is no continuous bijection $g : C \longrightarrow M$ between the cylinder C and the twisted cylinder M such that g sends the d_C-distance between any two points $P, Q \in C$, equals the d_M-distance between $g(P), g(Q) \in M$.

Solution.

(a) These distances are found by plotting one point, drawing the glides of the other, and noticing the closest approach. Notice that S and T are the same point, because

$$(0.9, 0.9) = (-0.1 + 1, (-1)^1(-0.9)).$$

We find

$$\begin{aligned}
d_M(P,Q) &= \sqrt{(0.2)^2 + (0.1)^2} = \sqrt{0.05} \approx 0.224 \\
d_M(P,R) &= \sqrt{0^2 + (0.9)^2} = 0.9 \\
d_M(P,S) &= d_M(P,T) = \sqrt{(0.2)^2 + (0.9)^2} = \sqrt{0.85} \approx 0.922 \\
d_M(Q,R) &= \sqrt{(0.2)^2 + (0.8)^2} = \sqrt{0.68} \approx 0.625 \\
d_M(Q,S) &= d_M(Q,T) = \sqrt{0^2 + (0.8)^2} = 0.8 \\
d_M(R,S) &= d_M(R,T) = \sqrt{(0.8)^2 + 0^2} = 0.8 \\
d_M(S,T) &= 0.
\end{aligned}$$

Remark: Notice that $d_M(P,Q) = 0$ if and only if $P = Q$ in M. Any function deserving of the name "distance" *must* have this property, along with the other properties mentioned in Homework 1.

(b) A straightforard technique here is the following, propagate all copies of L and K in $\mathbb{R}^2$ under the action of the group Γ. For L, this involves all horizontal translations by even integers, and all "flipped" lines translated horizontally by odd integers. For K this is just all integer horizontal transformations. Then look at the set of points in $\mathbb{R}^2$ where these propagations cross each other (not the crossings of L with itself though, etc.). Project this infinite set of crossings to M to see the actual points of crossing on the torus. In this case, all crossings of $\Gamma(K)$ with $\Gamma(L)$ occur at the places where $\Gamma(L)$ crosses itself.

We find that the intersection points are

$$\pi(K) \cap \pi(N) = \langle t_{(1,1)}, t_{(1,-1)} \rangle \, (0.5, 0.5) = \{(0.5 + n + m, 0.5 + n - m) : (n,m) \in \mathbb{Z}^2\}.$$

More rigorously our sets are

$$\pi(L) = \bigcup_{m \in \mathbb{Z}} \{x = 0.5 + m\}$$

and

$$\pi(K) = \left(\bigcup_{n \in \mathbb{Z}} \{y = x + 2n\} \right) \cup \left(\bigcup_{n \in \mathbb{Z}} \{y = -x + 2n + 1\} \right).$$

We intersect $\pi(L)$ with each of the two collections in $\pi(K)$ separately. First, solving

$$x = 0.5 + m$$
$$y = x + 2n$$

gives the set of points

$$\{(0.5 + m, 0.5 + m + 2n : (m, n) \in \mathbb{Z}^2\},$$

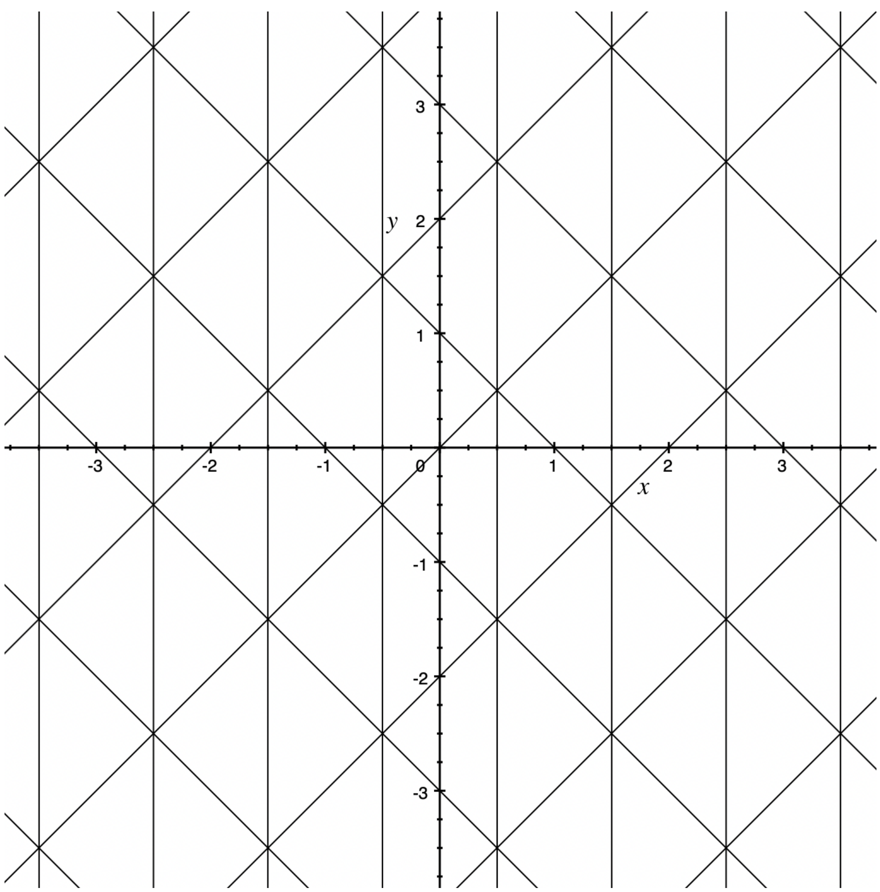
and solving

$$x = 0.5 + m$$
$$y = -x + 2n + 1$$

gives the set of points

$$\{(0.5 + m, 0.5 - m + 2m : (m, n) \in \mathbb{Z}^2\}.$$

These two sets are actually *equal*, and both are equal to the set found earlier (make sure you see why). The fact that they are equal can be seen by the picture below, where the vertical lines cross both types of diagonals in all the same places.



We can rewrite this set as

$$\langle t_{(1,1)}, t_{(1,-1)} \rangle (0.5, 0.5) = \{(0.5 + n, 0.5 + m) : (n, m) \in \mathbb{Z}^2, n + m \text{ is even}\},$$

Which is a little easier to manipulate. The condition that $n + m$ is even is the same as saying that $n - m$ is even, which is just to say that n and m are either both odd or both even. In any case, we now have to project this set to M. In symbols, we are performing the quotient

$$\langle t_{(1,1)}, t_{(1,-1)}\rangle\,(0.5, 0.5)/\,\langle t_{(1,0)} \circ \overline{r}\rangle\,.$$

Take $n, m \in \mathbb{Z}^2$ both even or both odd. For all $k \in \mathbb{Z}$, we want to identify the points

$$(0.5 + n, 0.5 + m) \sim (0.5 + n + k, (-1)^k(0.5 + m))$$

if they both occur in the set $I := \langle t_{(1,1)}, t_{(1,-1)}\rangle\,(0.5, 0.5)$. If k is even, then we have

$$(0.5 + n, 0.5 + m) \sim (0.5 + n + k, 0.5 + m),$$

so we are identifying all points which differ by an even horizontal shift. Therefore, we can represent I by just two horizontal values, $n = 0$ or $n = -1$. That is, points of the form

$$\begin{array}{lll} (0.5, 0.5 + m) & \text{for even } m, & \text{and} \\ (-0.5, 0.5 + m) & \text{for odd } m. & \end{array}$$

The latter set may also be described as $(-0.5, -0.5 + m)$ for even m. So we are looking at vertical translations by even integers of the points $(0.5, 0.5)$ and $(-0.5, -0.5)$. Call these the right and left columns, respectively.

Finally, we identify points using the odd values of k. For the right column, we have

$$(0.5, 0.5 + m) \sim (0.5 + k, -0.5 - m)) = (-0.5, -0.5 - m),$$

where in the last equality we used the identification just observed. This identifies every point in the left column with a point (on the other side of the x-axis) in the right column. Applying the identification to the left column yields the same relation. Therefore, after all identifications have been made, we see that

$$I = \{(0.5, 0.5 + 2m) : m \in \mathbb{Z}\} \subseteq M.$$

is the minimal set of representatives for our crossings. All even vertical translations of a single point.

Remark Try to deduce this result straight from the picture shown, without all the fancy symbols. Each element in the final version of I is acted upon by Γ to produce glide images, tracing out two lines horizontally.

(c) Play the same game as in part (b). Propagating the lines in $\mathbb{R}^2$, we find a crossing at every integer lattice point (that is, $\{(n, m) \in \mathbb{Z}^2\}$). Make sure not to count *self*-crossings of either K or N. Quotienting, we find that each lattice point on the y-axis is a representative. The set of crossings in M is therefore

$$\{(0, n) : n \in \mathbb{Z}\}$$

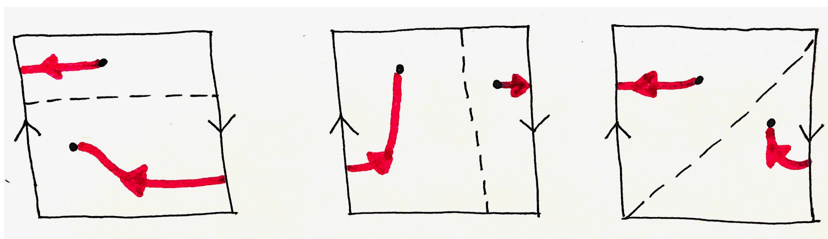
and it is countably infinite in number.

(d) As in the cylinder, vertical lines work nicely. Take the lines $\{x = 0\}$ and $\{x = 0.5\}$. Drawing their propagations, you can see that they never intersect in $\mathbb{R}^2$.

Remark: Unlike the cylinder, however, not all parallel lines in $\mathbb{R}^2$ descend to parallel lines in M. Take the lines $\{y = x\}$ and $\{y = x + 0.5\}$ in $\mathbb{R}^2$. Draw their propagations to see that they intersect in M.

(e) Yes. Take L_1 and L_2 to be the projections of the x- and y-axes, respectively, to M. Propagating the x- and y-axes in $\mathbb{R}^2$ according to the action by Γ, we see that the only crossings are $\{(n, 0) : n \in \mathbb{Z}\}$, integer points on the x-axis. Under the quotient, these all become identified, so the origin $(0, 0) \in M$ is the single crossing of L_1 and L_2.

(f) The point is that, if g exists, then it would basically take all properties of C to M, and vice versa, so it would tell us that C and M are essentially the same, which shouldn't be true. The main idea in the argument is this: if you draw a horizontal line in C, it splits C into two disjoint pieces, but no line in M can split M into two pieces. See the figure below, where we show that M cannot be split by any of the three types of lines in M: horizontal, vertical, and diagonal.



More precisely, suppose a continuous bijection $g : C \longrightarrow M$ exists. Let L be the line $\{x = 0\}$ in C. Let P be a point above L, and Q a point below L. Since L cuts C into two pieces, there is no path we can draw between P and Q that doesn't cross through M. Applying g, we now have two points $g(P)$ and $g(Q)$, and a line $g(L)$ in M (an isometry will map lines to lines). By the figure above, we can always find a path S in M from $g(P)$ to $g(Q)$ that avoids $g(L)$. But then if we apply g^{-1} to everything, we will have a path $g^{-1}(S)$ from P to Q which avoids L (because g is bijective), a contradiction.

Problem 5. (20 pts) (**The Torus**) Let $T^2 = \mathbb{R}^2/\Gamma$ be the Euclidean Torus, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by the two translations

$$t_{(0,1)}, t_{(1,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

We use coordinates $(x, y) \in T^2$, induced by the coordinates $(x, y) \in \mathbb{R}^2$, where in the torus T^2 we identify $(x, y) \sim (x + n, y + m)$ for any $(n, m) \in \mathbb{Z}^2$. Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the torus T^2 is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) Find the distance between the two points $(1.1, -12), (-1.9, 23.8) \in T^2$.

(b) Let $L_\alpha = \{x = \alpha y\} \subseteq \mathbb{R}^2$ be the line of slope α. Suppose α is a non-zero rational number. Show that the intersection of $\pi(L_\alpha)$ and $\pi(L_0)$ consists of finitely many points.

(c) Assume α is a non-zero irrational number. How many times do the lines $\pi(L_\alpha)$ and $\pi(L_0)$ intersect ?

(d) Find four lines $L_1, L_2, L_3, L_4 \subseteq T^2$ such that $|L_i \cap L_j| = \emptyset$ for $1 \leq i, j \leq 4$, $i \neq j$, i.e. four lines which they do *not* pairwise intersect.

(e) For any $\alpha \in \mathbb{R}$, find infinitely many lines $L \subseteq \mathbb{R}^2$ such that $\pi(L) = L_\alpha$.

Solution.

(a) This is like the cylinder, except that we do the reduction to both coordinates.

$$d_{T^2}((1.1, -12), (-1.9, 23.8)) = \sqrt{0^2 + (0.2)^2} = 0.2.$$

(b) As in Problem 4, it is good to draw a picture here. Since the fundamental domain of T^2 is a square S of side length 1, the heuristic idea here is to trace L_α through S, starting back on one side when you hit a wall, and eventually connecting back to the origin (since α is rational). The number of times you "hit" a vertical wall is the number of crossings. If you think of the torus in $\mathbb{R}^3$, you should think of L_0 as wrapping around the doughnut, while L_0 just hangs out as a belt all the way around.

Rigorously, first notice that

$$\pi(L_\alpha) = \bigcup_{(n,m)\in\mathbb{Z}^2} \{x - n = \alpha(y - m)\}.$$

In particular, $\pi(L_0)$ is all vertical lines at integer values of x. Intersecting, we need to solve

$$x - k = 0$$
$$x - n = \alpha(y - m)$$

for all integers $k, n, m \in \mathbb{Z}$. This gives

$$\pi(L_0) \cap \pi(L_\alpha) = \{(k, m + \tfrac{1}{\alpha}(k - n)) : (k, n, m) \in \mathbb{Z}^3\}.$$

Already we can identify many of these points. For any k and m, we can move through the integer lattice to reach an equivalent point $(0, \frac{1}{\alpha}(k - n))$, so we're really just dealing with the set

$$\{(0, \tfrac{1}{\alpha}(k - n)) : (k, n) \in \mathbb{Z}^2\} = \{(0, \tfrac{n}{\alpha}) : n \in \mathbb{Z}\}.$$

Since α is a rational number, represent it in lowest terms as $\alpha = \frac{p}{q}$ for integers p and q, with $p > 0$. Then our set is

$$\{(0, \tfrac{q}{p}n : n \in \mathbb{Z}\} = \{(0, \tfrac{q}{p}n) : n \in \{0, 1, \dots, p - 1\}\},$$

because $(0, \frac{q}{p} \cdot p) = (0, q) = (0, 0)$ in the torus. Therefore, there are exactly p intersections.

(c) If α is irrational, then $\frac{n}{\alpha}$ is not an integer for any n. Furthermore, for two integers n and m, $\frac{n}{\alpha} - \frac{m}{\alpha} = \frac{1}{\alpha}(n - m)$ is also never an integer. Therefore, no two points $(0, \frac{n}{\alpha})$ and $(0, \frac{m}{\alpha})$ are equivalent in T^2, and we see that our intersecting set

$$\{(0, \tfrac{n}{\alpha}) : n \in \mathbb{Z}\}$$

can't be simplified anymore. There are infinite intersections.

(d) Take any four parallel lines in $\mathbb{R}^2$ that do not become identified exactly in T^2 (so, no two of them are integer translations of each other). These descend to parallel lines in T^2 just like they do in the cylinder C.

Remark: You get some spirals (some are closed, some aren't), some circles that go through the doughnut hole, and some circles that go around the whole doughnut. The spirals are diagonal lines, and the two types of circles are horizontals and verticals, depending on in what order you glued your torus together. The doughnut hole circles come from the first identification, which are the horizontals in C, and the wrap-arounds come from the second identification, which were the verticals in C. But you could choose the other order, and your first identification would look like a different version of C than the one we usually discuss.

(e) There is a "doubly"-infinite set of lines. We described them in the solution to (b). They create the family $\{L^{(n,m)} : (n, m) \in \mathbb{Z}^2\}$, where $L^{(n,m)} \subseteq \mathbb{R}^2$ is the line $\{x - n = \alpha(y - m)\}$.

Problem 6. (20 pts) (**The Klein Bottle**) Let $K = \mathbb{R}^2/\Gamma$ be the Klein bottle, where $\Gamma = \langle t_{(0,1)}, t_{(1,0)} \circ \overline{r}\rangle \subseteq \mathrm{Iso}(\mathbb{R}^2)$ is the group generated by a translation and a glide reflection. We use coordinates $(x, y) \in K$, induced by the coordinates $(x, y) \in \mathbb{R}^2$, where in the Klein bottle K we identify $(x, y) \sim (x + n, (-1)^n y)$ and also $(x, y) \sim (x, y + n)$, for any $n \in \mathbb{Z}$. Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ be the projection map, sending a point to its Γ-orbit. By definition, a line in the Klein Bottle is the image of a line $L \subseteq \mathbb{R}^2$ under the map π.

(a) Draw the Γ-orbits $\Gamma(P), \Gamma(Q), \Gamma(R) \subseteq \mathbb{R}^2$ for each of the three points $P = (0, 0)$ and $Q = (1, 0)$ and $R = (-1, 0)$ in the Euclidean plane $\mathbb{R}^2$.

(b) Find the distance between the two points $(0, 0), (0.9, 0.9) \in K$.

(c) Show that there exist parallel lines inside a Klein Bottle.

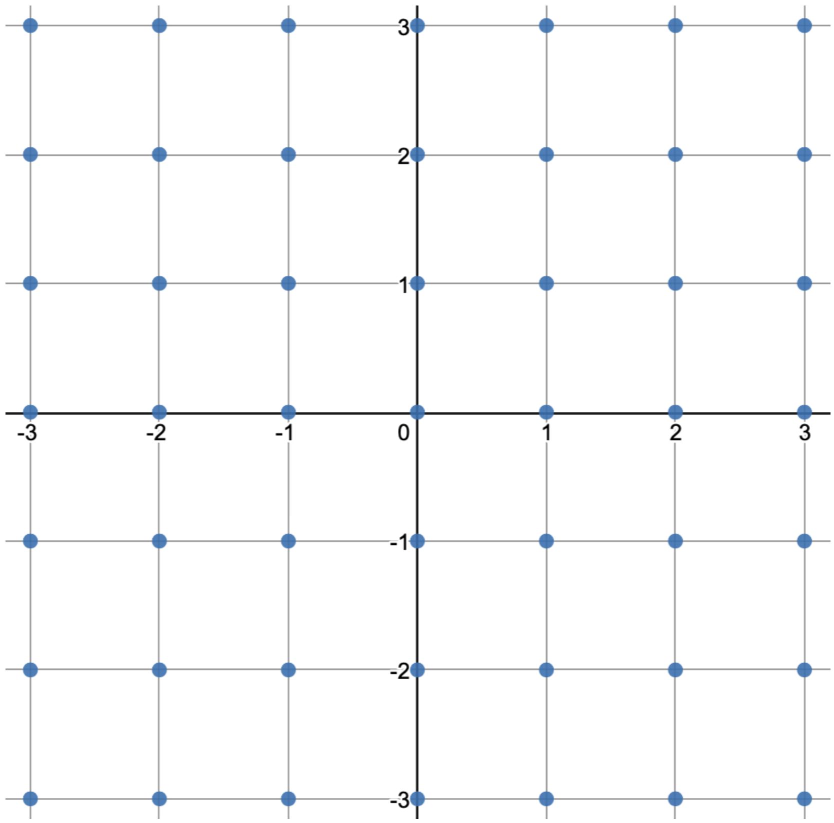
(d) Consider the lines $L = \{y = 0\} \subseteq \mathbb{R}^2$ and $W = \{x = y\} \subseteq \mathbb{R}^2$. Find the intersection points of $\pi(L)$ and $\pi(W)$.

(e) Let $N = \{(x, y) \in \mathbb{R}^2 : y = 0.5\} \subseteq \mathbb{R}^2$. Find the number of intersection points of $\pi(W)$ and $\pi(N)$.

(f) Show that the Klein bottle can be cut into two Möbius bands.

Solution.

(a) All three points have the same Γ-orbit, the integer lattice $\mathbb{Z}^2$.



In general, finding the Γ-orbit of a general point P in K might seem tricky. In the cylinder and twisted cylinder, our subgroup Γ of isometries had only one generator. In the torus, Γ was generated by two commuting elements, so we could think of any group element as first some horizontal translation, and then some vertical translation.

In the Klein bottle, our generators don't commute, but they *almost* do, in the following way: let $t := t_{(0,1)}$ and $g := t_{(1,0)} \circ \overline{r}$ represent the translation and glide generators of Γ, respectively. They don't commute, but you can check by calculation that we do have

$$g^m \circ t^n = t^{(-1)^m n} \circ g^m$$

for any integers n and m. So, if we have a general group element with a bunch of t's and g's in it, we can always keep moving the g's to the right, just changing the t's as we go. Therefore, we can say confidently that

$$\Gamma = \langle t, g \rangle = \{t^n g^m : (n, m) \in \mathbb{Z}^2\}.$$

(We could have also written it the other way around). Hence, similar to the torus, we can visualize any element of $\Gamma(P)$ as some horizontal propagation by a power of g, *followed* by a propagation vertically by powers of t. Therefore, to visualize $\Gamma(P)$, simply trace out some glides left and right, and then trace those glide lines up and down.

(b) In light of part (a) and Problem 1, we just need to find the distance from $(0.9, 0.9)$ to its closest lattice point, which is $(1, 1)$. Therefore,

$$d_K((0,0),(0.9,0.9)) = d((1,1),(0.9,0.9)) = \sqrt{0.02} \approx 0.141.$$

(c) By the nature of quotienting, lines in $\mathbb{R}^2$ which intersect will never become parallel in the quotient, though parallel lines in $\mathbb{R}^2$ may end up intersecting once quotiented. Therefore, we are led (as always) to think about *which* parallel lines in $\mathbb{R}^2$ remain parallel in the quotient $\pi : \mathbb{R}^2 \longrightarrow K$.

As usual, close vertical lines are our friends. The lines $\{x = 0\}$ and $\{x = 0.5\}$ descend in K to the families $\{\{x = n\} : n \in \mathbb{Z}\}$ and $\{\{x = 0.5 + n\} : n \in \mathbb{Z}\}$ respectively, which certainly do not intersect. Similar horizontal lines work too.

(d) Convince yourself (using equations like Problem 5(b), or just pictures) that

$$\pi(L) = \bigcup_{m\in\mathbb{Z}} \{y = m\},$$

and

$$\pi(W) = \left(\bigcup_{n\in\mathbb{Z}} \{y = x + n\}\right) \cup \left(\bigcup_{n\in\mathbb{Z}} \{y = -x + n\}\right).$$

Either geometrically or algebraically, we find that the set of intersections is

$$\{\pi(L) \cap \pi(W) = \{(n, m) : (n, m) \in \mathbb{Z}^2\},$$

the whole integer lattice (make sure not to count any intersections of $\pi(W)$ with *itself*). But we showed in part (a) that these points are all the same, so $\pi(L)$ and $\pi(W)$ intersect only at the origin $(0, 0) \in K$.

(e) The only difference here is that

$$\pi(N) = \bigcup_{m\in\mathbb{Z}} \{y = 0.5 + m\}.$$

Now our intersection set is

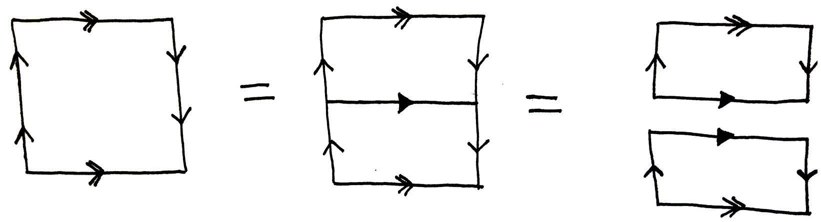
$$\{\pi(N) \cap \pi(W) = \{(n + 0.5, m + 0.5) : (n, m) \in \mathbb{Z}^2.$$

Interestingly, these are the remaining self-crossings of $\pi(W)$ other than those found in the intersection from part (d). As before, these are all the same point, so $\pi(W)$ an $\pi(L)$ cross in a single point $(0.5, 0.5) \in K$.

(f) The common way to work with these objects is with a *polygonal presentation*, which means we take a fundamental domain and describe the identifications along the boundary. We know that a fundamental domain for the Klein bottle is a square. We identify the boundaries by gluing the horizontal sides together with the identity, and gluing the vertical sides together with a flip.

For the twisted cylinder, a fundamental domain is an infinite strip, but when we call it a Möbius band, we mean to just chop off a horizontal rectangle in that strip. Then the only identification is gluing the vertical sides together with a flip.

Shown below is a Klein bottle. We “cut” along the middle line to produce two Möbius bands. Note that within each particular Möbius band, the arrows on the horizontal edges are not to be identified. Instead, the two bands glue together along their horizontal edges in the way indicated by the shadings.



Problem 7. (20 pts) For each of the ten sentences below, justify whether they are **true** or **false**. If true, you must provide a proof, if false you must provide a counter-example.

(a) For each line $\mathcal{L} \subseteq C$ in the cylinder, there are infinitely many lines $L \subseteq \mathbb{R}^2$ such that $\pi(L) = \mathcal{L}$.

(b) For each point $p \subseteq C$ in the cylinder, there are infinitely many points $P \subseteq \mathbb{R}^2$ such that $\pi(P) = p$.

(c) For each line $\mathcal{L} \subseteq T^2$ in the 2-torus, there are infinitely many lines $L \subseteq \mathbb{R}^2$ such that $\pi(L) = \mathcal{L}$.

(d) The only isometry of the twisted cylinder M is the identity.

(e) Any isometry of the cylinder C must be a vertical translation, i.e. an integer multiple of the isometry $t_{(0,1)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, descended to $t_{(0,1)} : \mathbb{R}^2/\Gamma \longrightarrow \mathbb{R}^2/\Gamma$.

(f) An isometry $f : T^2 \longrightarrow T^2$ is uniquely determined by the three images

$$f(A), f(B), f(C) \in T^2$$

of three non-collinear points $A, B, C \in T^2$.

(g) Let $P, Q \in C$ be two points in the cylinder. Then there exists an isometry $f : C \longrightarrow C$ such that $f(P) = Q$.

(h) Let $P, Q \in T^2$ be two points in the torus. Then there exists an isometry $f : T^2 \longrightarrow T^2$ such that $f(P) = Q$.

(i) Let $P, Q \in C$ be distinct, then the set of points $R \in C$ which are equidistant to P and Q form a line in C, i.e. $\{R \in C : d(R, P) = d(R, Q)\}$ is a line.

(j) Let $P, Q \in T^2$ be distinct, then the set of points $R \in T^2$ which are equidistant to P and Q form a line in T^2, i.e. $\{R \in T^2 : d(R, P) = d(R, Q)\}$ is a line.

Solution.

(a) False. This is true for verticals and spirals, but not for the horizontal circle in C. Every circle in C is the image under π of just a single horizontal line in $\mathbb{R}^2$. This is because translating a line L in $\mathbb{R}^2$ horizontally results in a new line if and only if L is not already horizontal.

(b) True. For any point P which represents p, any horizontal translation of P by an integer distance will also represent p.

(c) True. The issue that happened in part (a) for the cylinder doesn't happen here, because we have the symmetry of identifying in two directions. In Problem 5(e), we found an infinite family of lines that projected to any given line through the origin in T^2, but there's nothing special about the origin. The same procedure can be done for any line. Indeed, let $\mathcal{L} \subseteq T^2$ be the line $\pi(\{ay + bx = c\})$ for arbitrary $a, b, c \in \mathbb{R}$. Take the infinite family of lines $\{L^{(n,m)} : (n, m) \in \mathbb{Z}^2\}$, where

$$L^{(n,m)} = \{a(y - n) + b(x - m) = c\} \subseteq \mathbb{R}^2.$$

Then $\pi(L^{(n,m)}) = \mathcal{L}$ for all $(n, m) \in \mathbb{Z}^2$.

(d) False. Non-trivial horizontal translations also give us isometries. Here is the proof of why. Take a translation $t_{(\alpha,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. First we show that it is well-defined on M. Take $P \in \mathbb{R}^2$. Then its orbit $\Gamma(P)$ is in M, and the translation $t_{(\alpha,0)} : M \longrightarrow M$ acts by

$$t_{(\alpha,0)}(\Gamma(P)) = \Gamma(t_{(\alpha,0)}(P)).$$

to show that it is well-defined, we need to show that $t_{(\alpha,0)}(\Gamma(P)) = t_{(\alpha,0)}(\Gamma(P'))$ for any $P' \in \Gamma(P)$. Since Γ is generated by one element, we can assume that $P' = t_{(1,0)} \circ \overline{r}(P)$. Then

$$\begin{aligned} t_{(\alpha,0)}(\Gamma(P')) &= \Gamma(t_{(\alpha,0)}(P')) \\ &= \Gamma(t_{(1+\alpha,0)} \circ \overline{r}(P)) \\ &= \Gamma(\overline{r} \circ t_{(1+\alpha,0)}(P)) \\ &= \Gamma(\overline{r} \circ t_{(1,0)}(t_{(\alpha,0)}(P))) \\ &= t_{(\alpha,0)}(\Gamma(P)), \end{aligned}$$

Because $\overline{r} \circ t_{(1,0)}$ is a group element of Γ. Now we show that this is actually an isometry, using the fact that it is an isometry in the plane. For two points $\Gamma(P)$ and $\Gamma(Q)$ in M, we have

$$\begin{aligned} d_M(t_{(\alpha,0)}(\Gamma(P)), t_{(\alpha,0)}(\Gamma(Q))) &= d_M(\Gamma(t_{(\alpha,0)}(P)), \Gamma(t_{(\alpha,0)}(Q)) \\ &= \min\{d(t_{(\alpha,0)}(P), Q') : Q' \in \Gamma(t_{(\alpha,0)}(Q))\} \\ &= \min\{d(t_{(\alpha,0)}(P), (t_{(1,0)}\overline{r})^n(t_{(\alpha,0)}(Q))) : n \in \mathbb{Z}\} \\ &= \min\{d(t_{(\alpha,0)}(P), t_{(\alpha,0)}((t_{(1,0)}\overline{r})^n(Q))) : n \in \mathbb{Z}\} \\ &= \min\{d(P, (t_{(1,0)}\overline{r})^n(Q)) : n \in \mathbb{Z}\} \\ &= \min\{d(P, Q') : Q' \in \Gamma(Q)\} \\ &= d_M(\Gamma(P), \Gamma(Q)), \end{aligned}$$

so this is an isometry on M. And $t_{(\alpha,0)}$ is not the identity for any non-integer value of α. This is essentially a rotation of the Mobiüs strip.

Remark: Note that vertical translations $t_{(0,\beta)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ don't even descend to the twisted cylinder (meaning they aren't well-defined on equivalence classes) because they don't respect the reflection nature of the glide which generates the group Γ. But horizontal translations do. This is due to the phenomenon that a translation commutes with a glide if and only if they are in the same direction.

(e) False. The example given in the solution to part (d) above works just as well on the un-twisted cylinder. Please make sure you're able to adapt the proof given to this case.

(f) False. The key part of the proof of this fact in the Euclidean plane was that the set of points equidistant from any two given points is a line. There, for any point P, $f(P)$ was uniquely determined by its distances from $f(A)$, $f(B)$, and $f(C)$ (which are the same as those from P to A, B, and C, because f is an isometry). This was because if there were a second point Q that had the same distances from these three images as did $f(P)$, then $f(A)$, $f(B)$, and $f(C)$ would lie in the set of points equidistant from $f(P)$ and Q, meaning they would be collinear.

This argument fails because of part (j) below. There we see that the set of points equidistant from two points may be *two* lines, which suggests that we take A, B, and C on these two lines. Indeed, let S be the usual fundamental domain of T^2, a square with side length 1, corner the origin, in the first quadrant. Let $A = (0, 0)$, $B = (0.5, 0)$, and $C = (0.5, 0.5)$. These three points are clearly non-collinear.

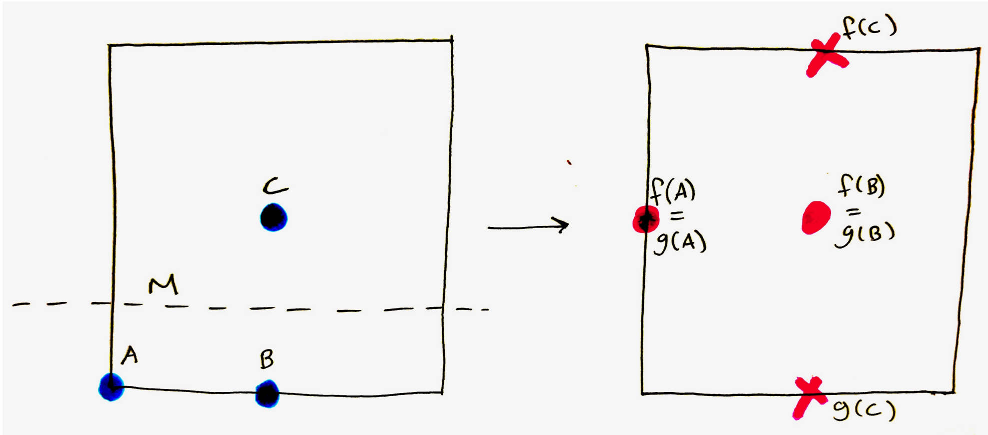
Next, notice that in $\mathbb{R}^2$, vertical reflections, horizontal reflections, and any translation all descend to well-defined isometries of T^2. In particular, notice that if two points in $\mathbb{R}^2$ differ by an integer lattice translation, then so do their images under an isometry of one of these types. Therefore, it makes sense to define $f : T^2 \longrightarrow T^2$ and $g : T^2 \longrightarrow T^2$ by

$$f = t_{(0,0.5)} \qquad \text{and} \qquad g = \overline{r}_M,$$

where M is the line $\{y = 0.25\}$ in the plane. Notice that these are *different* isometries in T^2. In particular, g fixes a line, but f does not. However,

$$\begin{aligned} f(A) &= (0.0.5) = g(A) \\ f(B) &= (0.5, 0.5) = g(B), \qquad \text{and} \\ f(C) &= (0.5, 1) = (0.5, 0) = g(C). \end{aligned}$$

So we have two different isometries with the same image on three non-collinear points. The essential part was using the equivalence of two points in the last line. If we were looking at representatives of A, B, and C in $\mathbb{R}^2$, then f and g clearly do not agree, but they agree *in the quotient*.



(g) True. Take representatives of P and Q in $\mathbb{R}^2$ and apply the translation that sends one to the other. This translation descends to an isometry in the cylinder. (Visually, this looks like a rotation of the cylinder composed with a shift up or down.)

(h) True. Apply the same argument from the solution to (g) above. Now this translation looks like two different rotations in a row, one where you spin the doughnut around its axis, and another where the donut rotates in around itself.

(i) False. Take $P = (0, 0)$ and $Q = (0.5, 0)$ in the cylinder. Then the set of points which are equidistant from P and Q is formed by two lines L and L', the projections of the lines $\{x = 0.25\}$ and $\{x = 0.75\}$ in $\mathbb{R}^2$. In fact, this will happen whenever P and Q are two distinct points with the same y-coordinate.

(j) False. The same example given in the solution to (i) above works in the torus too. The two vertical linens $\{x = 0.25\}$ and $\{x = 0.75\}$ now become two closed circles, but they are not identified because they differ by a non-integer translation.