SOLUTIONS TO PROBLEM SET 4

MAT 141

Abstract. These are the solutions to Problem Set 4 for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. The problems were posted online on Saturday Feb 15 and due Friday Feb 21 at 10:00am.

Problem 1. Decide whether the following points $P \in \mathbb{R}^3$ belong to the 2-sphere S^2:

$$(1,0,0),(0,1,0),(1,1,0),(1/\sqrt{2},1/\sqrt{2},0),$$
$$(1/2,0,1/2),(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}),(1/4,1/2,1/2).$$

Solution. Remember that the definition of the 2-sphere is

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\} \subseteq \mathbb{R}^3,$$

the set of all points in $\mathbb{R}^3$ with norm 1. Therefore, to figure out whether a point is in S^2, we just need to calculate its norm. We find that

$$(1,0,0),(0,1,0),(1/\sqrt{2},1/\sqrt{2},0), \quad \text{and} \quad (1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})$$

all have norm 1, so these are in S^2. The rest of the points given do not have norm 1, so they are not in S^2.

Problem 2. For each pair of axis $l_1, l_2 \subseteq \mathbb{R}^3$, find a *linear* isometry $\varphi \in \text{Iso}(\mathbb{R}^3)$, $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, such that $\varphi(l_1) = l_2$.

(a) Let $l_1 = \langle(1,0,0)\rangle$ be the oriented axis spanned by the vector $(1,0,0)$, and choose $l_2 = \langle(0,0,1)\rangle$.

(b) Let $l_1 = \langle(0,1,0)\rangle$ and choose $l_2 = \langle(0,0,1)\rangle$.

(c) Let $l_1 = \langle(1,1,0)\rangle$ and choose $l_2 = \langle(0,0,1)\rangle$.

(d) Let $l_1 = \langle(1,1,2)\rangle$ and choose $l_2 = \langle(0,0,1)\rangle$.

Solution. This is essentially what we did in Discussion 6, constructing the matrix that rotates a given point P to the point $(0,0,1)$ on the z-axis. Remember, we first perform a rotation $R_{z,\phi}$ to bring P to the yz-plane (to make $x = 0$), and then we perform a rotation $R_{x,\psi}$ to bring $P' = R_{z,\varphi}(P)$ to the point $(0,0,1)$ on the z-axis. We'll need to use the formulas

$$R_{z,\phi} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad R_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}.$$

Then, our transformation will be $\varphi = R_{x,\psi} \circ R_{z,\phi}$. Since φ is a composition of rotations at the origin, it is guaranteed to be a linear isometry of $\mathbb{R}^3$. (Note, in the problems below, you should always check that $\varphi(P) = (0,0,1)$ and that φ is an orthogonal matrix.)

Remark. Be aware that the answer φ is not unique. Once you've used φ to rotate P to some given point Q (here something on the positive z-axis), any further rotation γ about Q will produce a new isometry that fixes Q. Forming the composition $\gamma \circ \varphi$ will produce a linear isometry just as valid as the original φ. (Really, any 3×3 orthogonal matrix will due, as long as it sends P to Q).

(a) This is really just rotation about the y-axis, but it will follow systematically from our procedure described above. To rotate $P = (1,0,0)$ into the yz-plane, we perform $R_{z,\frac{\pi}{2}}$, resulting in

$$R_{z,\frac{\pi}{2}}(P) = (0,1,0).$$

For the second stage, use part (b) below, which rotates $(0,1,0)$ to $(0,0,1)$. Therefore,

$$\varphi = R_{x,\frac{\pi}{2}} \circ R_{z,\frac{\pi}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Remark. We could also have found a valid isometry more geometrically. The columns of our matrix could be given by the images of the basis vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ under a rotation about the y-axis by $\frac{\pi}{2}$. We definitely want $(1,0,0) \mapsto (0,0,1)$. Since we are rotating about the y-axis, we can choose $(0,1,0) \mapsto (0,1,0)$. Finally, by imagining the picture, we see that $(0,0,1) \mapsto (-1,0,0)$. Therefore, our rotation is

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

another valid answer.

(b) Since $P = (0,1,0)$ is already in the yz-plane, we only need to rotate P into the z axis. Draw the yz-plane to see that the rotation $R_{x,\frac{\pi}{2}}$ will do the trick. Therefore,

$$\varphi = R_{x,\frac{\pi}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(c) Here, $P = (1,1,0)$ isn't a point on the sphere, so it can't be sent to $(0,0,1)$. Still, we can rotate P to the positive z-axis, and it should land at $(0,0,\sqrt{2})$. Alternatively, we could normalize P (multiply it by the reciprocal of its norm) so that it lands in the sphere, and then proceed as before. It really makes no difference either way, so let's stick to the form given. Draw the point P in the xy-plane to see that we need $R_{z,\frac{\pi}{4}}$ to bring P to the yz-plane. We end up with

$R_{z,\frac{\pi}{4}}(P) = (0, \sqrt{2}, 0)$, which is on the same ray from the origin as the point from (b). Therefore, we rotate to the positive z-axis as before. We have

$$\varphi = R_{x,\frac{\pi}{2}} \circ R_{z,\frac{\pi}{4}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

(d) This one starts out like (c). The projection of $P = (1, 1, 2)$ to the xy-plane is $(1, 1, 0)$, so we use $R_{z,\frac{\pi}{4}}$ again to rotate to the yz-plane. The rest is a little harder to visualize than the others, so we recall our algorithm. We have

$$P' = R_{z,\frac{\pi}{4}}(P) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ 2 \end{bmatrix}.$$

In the yz-plane, the vector P' makes an angle of $\arctan\left(\frac{-\sqrt{2}}{2}\right)$ with the positive z-axis, so we perform $R_{x,\psi}$, where $\psi = -\arctan\left(\frac{-\sqrt{2}}{2}\right) = \arctan\left(\frac{1}{\sqrt{2}}\right)$. Therefore, our isometry is

$$\varphi = R_{x,\,\arctan\left(1/\sqrt{2}\right)} \circ R_{z,\frac{\pi}{4}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix}.$$

Problem 3. (20 pts) **Distances in S^2**. Let (S^2, d_{S^2}) be the 2-sphere with its distance function d_{S^2}. We consider the set $S^2 \subseteq \mathbb{R}^2$ as the set of points

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and thus use Cartesian coordinates $(x, y, z) \in S^2$ for points in S^2.

(a) Compute the distance $d_{S^2}(P, Q)$ between $P = (0, 0, 1)$ and $Q = (0, 1, 0)$.

(b) Compute the distance between $(1/\sqrt{2}, 1/\sqrt{2}, 0) \in S^2$ and $P = (0, 0, 1)$.

(c) Draw the following four sets in the S^2:

$$E_{\pi/4} = \{R \in S^2 : d_{S^2}(P, R) = \pi/4\}, \quad E_{\pi/2} = \{R \in S^2 : d_{S^2}(P, R) = \pi/2\},$$
$$E_{3\pi/4} = \{R \in S^2 : d_{S^2}(P, R) = 3\pi/4\}, \quad E_{\pi} = \{R \in S^2 : d_{S^2}(P, R) = \pi\}.$$

(d) Show that given any two points $P_1, P_2 \in S^2$, we have the equality

$$\{R \in S^2 : d_{S^2}(P_1, R) = d_{S^2}(P_2, R)\} = \{R \in \mathbb{R}^3 : d_{\mathbb{R}^3}(P_1, R) = d_{\mathbb{R}^3}(P_2, R)\} \cap S^2,$$

comparing the sets of equidistant points R to P_1, P_2 in S^2 and $\mathbb{R}^3$.

Solution.

(a) Recall the definition: $d_{S^2}(P,Q) = 2\arcsin\big(\frac{1}{2}d_{\mathbb{R}^3}(P,Q)\big)$. Here, the Euclidean distance is $\sqrt{1^2+1^2+0^2} = \sqrt{2}$, so

$$d_{S^2}(P,Q) = 2\arcsin\left(\tfrac{1}{2}\sqrt{2}\right) = \frac{\pi}{2}.$$

This makes sense, because these two points lie on the standard equator $\{x = 0\}$ in S^2, and they make a right angle with the origin.

(b) The Euclidean distance is again $\sqrt{\frac{1}{2}+\frac{1}{2}+1} = \sqrt{2}$, so

$$d_{S^2}(P,Q) = \frac{\pi}{2}.$$

Therefore, these two points also create a right angle with the origin.

Remark. Note that any point in the xy-plane makes a right angle when connected to any point on the z-axis through the origin. In full generality, $d_{S^2}(P,Q)$ will be $\frac{\pi}{2}$ if and only if Q lies on an axis normal to some plane Π through the origin which contains P. Since there are many such planes Π, there are of course many points Q which have distance $\frac{\pi}{2}$ from a given point P (as we see in (c) below).

(c) These are *lines of latitude* If you draw the sphere with $(0,0,1)$ at the north pole, then $E_{\pi/2}$ is the equator (at latitude $0°$), $E_{\pi/4}$ and $E_{3\pi/4}$ are the horizontal circles at latitudes $45°$ and $-45°$, respectively, and $E_\pi = \{(0,0,-1)\}$ is the single point at the south pole.

Remark. The greatest distance between two points in S^2 is π. Furthermore, for any single point P, the set of points at distance π from P is just the single point $-P$ antipodal to P. For any other distance $d \in (0,\pi)$, the set of points distance d from P forms a circle on the sphere (only a great circle if $d = \pi/2$). When P is the North Pole, these circles are called "lines" of latitude, even though only one of them is technically a line on the sphere.

(d) We discussed this briefly in Discussion 6. The key is that arcsin is a one-to-one function (I marked that step with a $*$). We can peel back the layers of these sets:

$$\begin{aligned}
&\{R \in S^2 : d_{S^2}(P_1,R) = d_{S^2}(P_2,R)\} \\
&\qquad = \{R \in \mathbb{R}^3 : 2\arcsin\big(\tfrac{1}{2}d_{\mathbb{R}^3}(P_1,R)\big) = 2\arcsin\big(\tfrac{1}{2}d_{\mathbb{R}^3}(P_2,R)\big)\} \cap S^2 \\
&\qquad = \{R \in \mathbb{R}^3 : \arcsin\big(\tfrac{1}{2}d_{\mathbb{R}^3}(P_1,R)\big) = \arcsin\big(\tfrac{1}{2}d_{\mathbb{R}^3}(P_2,R)\big)\} \cap S^2 \\
&\qquad \overset{*}{=} \{R \in \mathbb{R}^3 : \tfrac{1}{2}d_{\mathbb{R}^3}(P_1,R)) = \tfrac{1}{2}d_{\mathbb{R}^3}(P_2,R))\} \cap S^2 \\
&\qquad = \{R \in \mathbb{R}^3 : d_{\mathbb{R}^3}(P_1,R)) = d_{\mathbb{R}^3}(P_2,R))\} \cap S^2
\end{aligned}$$

Remark This has an intuitive geometric interpretation. The left-hand side is the definition of a line in S^2, the set of points equidistant from two points P_1

and P_2. The right-hand-side (before the intersection) is the same, but allowing points in all of $\mathbb{R}^3$. This forms a plane, and intersecting with S^2 gives our familiar great circles. Therefore, we have proved that all lines in S^2 are great circles, and all great circles in S^2 are lines.

Problem 4. (20 pts) **Lines in S^2.** Let $L, M \subseteq S^2$ be two distinct lines. The map $a : S^2 \longrightarrow S^2$ defined by $(x, y, z) \longmapsto (-x, -y, -z)$ is called the antipodal map.

(a) Show that $L \cap M$ consists of exactly two distinct points P, Q.

(b) Let $L \cap M = \{P, Q\}$, show that $Q = a(P)$ and $P = a(Q)$, where $a : S^2 \longrightarrow S^2$ is the antipodal map.

(c) Show that for any line $L \subseteq S^2$ there exists a plane $\Pi_L \subseteq \mathbb{R}^3$ through the origin such that $L = \Pi_L \cap S^2$.

(d) Let $\widehat{\Pi} \subseteq \mathbb{R}^3$ be a 2-plane which does *not* contain the origin and such that the intersection $\widehat{\Pi} \cap S^2$ contains more than a point. Show that $\widehat{\Pi} \cap S^2$ must be a circle.

(e) In the same hypothesis of Part.(d), show that $\widehat{\Pi} \cap S^2$ is not a line in S^2.

Solution.

(a) From part (c) below, we know that

$$L \cap M = (\Pi_L \cap S^2) \cap (\Pi_M \cap S^2) = (\Pi_L \cap \Pi_M) \cap S^2.$$

Since L and M are distinct lines in S^2, Π_L and Π_M are distinct planes in $\mathbb{R}^3$. These planes cannot be parallel, because they both pass through the origin. Therefore, $\Pi_L \cap \Pi_M$ is a line in $\mathbb{R}^3$ passing through the origin. By a suitable isometry of S^2, we may assume that this line is the x-axis, which intersects S^2 at two points $R_1 = (1, 0, 0)$ and $R_2 = (-1, 0, 0)$. By Problem 5(c), our points remain antipodal through this isometry.

(b) From the proof of part (a), we saw that when $\Pi_L \cap \Pi_M$ is the x-axis, we have $R_2 = -R_1 = a(R_1)$, as desired. The other relation holds by noticing that $a \circ a = (-\operatorname{Id}) \circ (-\operatorname{Id}) = \operatorname{Id}$, and applying a to both sides of the equation $R_2 = a(R_1)$. The general case then follows from this one, because an isometry $\varphi : S^2 \longrightarrow S^2$ will bring the x-axis to any desired axis. After applying φ, we have $P = \varphi(R_1)$ and $Q = \varphi(R_2)$ as the two distinct points in $\varphi(\Pi_L) \cap \varphi(\Pi_M)$. Since $a = -\operatorname{Id}$, it commutes with all linear maps, so

$$a(Q) = a \circ \varphi(R_1) = \varphi \circ a(R_1) = \varphi(R_2) = P,$$

as desired (again, the other direction holds by applying a to both sides above).

(c) First, we prove the following: for any distinct points $P_1, P_2 \in \mathbb{R}^3$, the set of points equidistant to P_1 and P_2 is a plane. By a suitable isometry of $\mathbb{R}^3$, we can assume that our two points P_1 and P_2 lie on the x-axis, at points $P_1 = (-\alpha, 0, 0)$

and $P_2 = (\alpha, 0, 0)$. (Note that this isometry will almost certainly not preserve the sphere, but that's ok, because we're just proving a result about $\mathbb{R}^3$ right now.) Then we proceed as in the Lemma on p. 9 of Stillwell:

$$\begin{aligned}\{(x,y,z) \in \mathbb{R}^3 : d_{\mathbb{R}^3}((x,y,z), P_1) = d_{\mathbb{R}^3}((x,y,z), P_2)\}\\ &= \{(x,y,z) \in \mathbb{R}^3 : (x+\alpha)^2 + y^2 + z^2 = (x-\alpha)^2 + y^2 + z^2\}\\ &= \{(x,y,z) \in \mathbb{R}^3 : 2\alpha x = -2\alpha x\}\\ &= \{(x,y,z) \in \mathbb{R}^3 : x = 0\},\end{aligned}$$

where we used that $\alpha \neq 0$, because $P_1 \neq P_2$. The result is the yz-plane, so the claim holds: the set of points equidistant between any two distinct points in $\mathbb{R}^3$ is a plane.

Going back to the case at hand, we know that our line $L \subseteq S^2$ is the set of points equidistant from two distinct point $P, Q \in S^2$ (by definition). By Problem 3(d), this means that

$$L = \Pi_L \cap S^2,$$

where

$$\Pi_L = \{R \in \mathbb{R}^3 : d_{\mathbb{R}^3}(P,R)) = d_{\mathbb{R}^3}(Q,R))\} \cap S^2.$$

We have just proved that Π_L is a plane, since $P \neq Q$ are points in $\mathbb{R}^3$. Finally, letting $O = (0,0,0)$ be the origin of $\mathbb{R}^3$, we have

$$d_{\mathbb{R}^3}(P,O) = \|P\| = 1 = \|Q\| = d_{\mathbb{R}^3}(Q,O),$$

where we used the fact that P and Q are on the sphere, and so have norm 1. Therefore, the origin is in our plane Π_L, as desired.

(d) This result still holds even if $\widehat{\Pi}$ does contain the origin, because then we are in the situation of part (c). By a suitable isometry of S^2, we may assume that $\widehat{\Pi}$ is parallel to the xy-plane. That is,

$$\widehat{\Pi} = \{(x,y,z) \in \mathbb{R}^3 : z = r\},$$

for some $-1 < r < 1$ (any other value of r will either intersect the sphere at only one point, or not at all). Then

$$\begin{aligned}\widehat{\Pi} \cap S^2 &= \{(x,y,z) \in \mathbb{R}^3 : z = r \quad \text{and} \quad x^2 + y^2 + z^2 = 1\}\\ &= \{(x,y,z) \in \mathbb{R}^3 : z = r \quad \text{and} \quad x^2 + y^2 = 1 - r^2\}.\end{aligned}$$

Since $1 - r^2 > 0$, this is the intersection of a cylinder with a perpendicular plane, and is therefore a circle.

(e) Now let's go back to assuming that $\widehat{\Pi}$ does not contain the origin. If $\widehat{\Pi} \cap S^2$ were a line, say $L \subseteq S^2$, then we would have

$$\widehat{\Pi} \cap S^2 = L = \Pi_L \cap S^2$$

for some plane Π_L containing the origin (by part (c)). But Π_L intersects S^2 in a full circle (in particular, in three points not connected by a straight line in $\mathbb{R}^3$), so the plane Π_L is uniquely determined by its intersection with S^2. Therefore, $\widehat{\Pi} = \Pi_L$, a contradiction.

Problem 5. (20 pts) **The antipodal map in** S^2. Consider the antipodal map $a : S^2 \longrightarrow S^2$ defined by $(x, y, z) \longmapsto (-x, -y, -z)$.

(a) Show that a is an isometry of (S^2, d_{S^2}) and show that is has no fixed points.

(b) Express a as a composition of reflections in $\mathrm{Iso}(S^2)$.

(c) Let $f \in \mathrm{Iso}(S^2)$ be an isometry. Show that antipodal points P, Q remain antipodal after applying $f : S^2 \longrightarrow S^2$, i.e. prove that P, Q are antipodal if and only $f(P), f(Q)$ are antipodal.

Solution.

(a) Notice that $a = -\mathrm{Id}$, so it has no fixed points (because $P = -P$ implies $P = O$, the origin, which is not in S^2). The antipodal map is an isometry because $-\mathrm{Id}$ is an isometry of $\mathbb{R}^3$, and equality of distances in $\mathbb{R}^3$ imply equality of distances in S^2. Alternatively, each of the reflections in part (b) below is an isometry, so a is an isometry.

(b) The antipodal map is formed by negating three coordinates. If we negate each coordinate one at a time, each negation corresponds to reflection in a coordinate plane. For example $(x, y, z) \mapsto (x, -y, z)$ is reflection in the great circle formed by intersecting S^2 with the zx-plane. Therefore, let L_x, L_y, and L_z be the lines in S^2 formed by the intersections with planes in $\mathbb{R}^3$ as follows:

$$L_x = \{x = 0\} \cap S^2$$
$$L_y = \{y = 0\} \cap S^2$$
$$L_z = \{z = 0\} \cap S^2.$$

Then $a = \overline{r}_{L_z} \circ \overline{r}_{L_y} \circ \overline{r}_{L_x}$. Note that all three reflections commute, so we could have written this composition in any order.

(c) Since the inverse of an isometry is again an isometry, it suffices to prove one direction of the if and only if. Suppose P and Q are antipodal. Then $d_{S^2}(P, Q) = \pi$. Then, if f is an isometry of S^2, we have

$$d_{S^2}(f(P), f(Q)) = d_{S^2}(P, Q) = \pi.$$

Two points with distance π in the sphere must have distance 2 in $\mathbb{R}^3$. But since the sphere has radius 1, any two such points must be antipodal. Therefore, $f(P)$ and $f(Q)$ are antipodal.

Alternatively, we can use a similar reasoning from Problem 4(b). Since $a = -\mathrm{Id}$, it commutes with all linear maps. Therefore, if $a(P) = Q$, then

$$a(f(P)) = f(a(P)) = f(Q),$$

so $f(P)$ and $f(Q)$ are antipodal.

Problem 6. (20 pts) **Isometries in** S^2. Let $l \subseteq \mathbb{R}^3$ be the oriented axis in 3-space generated by the vector $v = (1, 1, 1)$.

(a) Express the rotation $R_{z,\pi/2}$ as a composition of two reflections.

(b) Where does $R_{z,\pi/2}$ send the point $\frac{1}{\sqrt{6}}(1, 1, 2) \in S^2$?

(b) Find a formula for the rotation $R_{l,\theta}$.

(c) Where does $R_{l,\theta}$ map the point $(0, 0, 1)$?

(d) Let $A \subseteq \mathbb{R}^3$ be any oriented axis. Show that a general rotation $R_{A,\theta}$ must have exactly two fixed points.

(e) In the hypothesis of Part (d), show that the two fixed points of a general rotation $R_{A,\theta}$ must be antipodal.

Solution.

(a) Thinking of $(0, 0, 1) \in S^2$ as the North Pole, $R_{z,\frac{\pi}{2}}$ represents rotation to the East by 90°. We can decompose this rotation as a composition of two reflections in lines which meet $(0, 0, 1)$ and form an oriented angle of $\frac{\pi}{4}$. To do so, we can choose any two lines of longitude satisfying these properties. Choose the lines

$$L_1 = \{x = 0\} \cap S^2 \qquad \text{and} \qquad L_2 = \{x + y = 0\} \cap S^2.$$

Notice that both of these lines contain the point $(0, 0, 1)$. Next, the normal vector for L_2 is a rotation by $\frac{\pi}{4}$ of the normal vector for L_1, so these two lines have the desired angle. Therefore,

(1)
$$R_{z,\frac{\pi}{2}} = \overline{r}_{L_2} \circ \overline{r}_{L_1}.$$

This can be checked by expressing $\overline{r}_{L_2}$ and $\overline{r}_{L_2}$ using conjugations of $\overline{r}_E$ by the isometries found in Problem 1(a) and 1(c), respectively. (Recall,

$$E = \{z = 0\} \cap S^2$$

is the standard equator in S^2, and $\overline{r}_E$ is the standard reflection, $(x, y, z) = (x, y, -z)$.) Alternatively, we can see directly what these reflections do. Since $\overline{r}_{L_1}$ is reflection through the yz-plane, we have $\overline{r}_{L_1}(x, y, z) = (-x, y, z)$. Since $\overline{r}_{L_2}$ is reflection through the plane $\{y = -x\}$, it preserves the z-coordinate, sends the positive x-axis to the negative y-axis, and send the positive y-axis to the negative x-axis. Therefore, $\overline{r}_{L_2}(x, y, z) = (-y, -x, z)$. So, in matrices, our decomposition (1) becomes

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which checks out in the matrix multiplication.

(b) We apply our rotation as a matrix:

$$R_{z,\frac{\pi}{2}}(\tfrac{1}{\sqrt{6}}(1,1,2)) = \frac{1}{\sqrt{6}}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

(b) We can proceed as in Problem 2. First, we find the isometry $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that sends $v = (1,1,1)$ to the positive z-axis. We will then conjugate the standard rotation $R_{z,\theta}$ by φ.

To start, we want to rotate v to the yz-plane, using a rotation $R_{z,\phi}$. Notice that the projection of v to the xy-plane is $(1,1)$, just like the point in Problem 2(c), where the necessary matrix was $R_{z,\frac{\pi}{4}}$. Performing this rotation, we have

$$R_{z,\frac{\pi}{4}}(v) = (0, \sqrt{2}, 1).$$

We want to use a rotation $R_{x,\psi}$ to rotate this point to the positive z-axis. In the yz-plane, the point $(\sqrt{2}, 1)$ makes an angle $\arctan(-\sqrt{2})$ with the positive z-axis, so we should use $\psi = -\arctan(-\sqrt{2}) = \arctan\sqrt{2}$. Performing these rotations, we have

$$\varphi = R_{x,\arctan\sqrt{2}} \circ R_{z,\frac{\pi}{4}} = \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}}\\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}.$$

We check that $\varphi(v) = (0,0,\sqrt{3})$ on the positive z-axis, as desired. We will also need

$$\begin{aligned}
\varphi^{-1} &= (R_{x,\arctan\sqrt{2}} \circ R_{z,\frac{\pi}{4}})^{-1}\\
&= (R_{z,\frac{\pi}{4}})^{-1} \circ (R_{x,\arctan\sqrt{2}})^{-1}\\
&= R_{z,-\frac{\pi}{4}} \circ R_{x,-\arctan\sqrt{2}}\\
&= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}.
\end{aligned}$$

Putting it all together, rotation about the oriented axis l by angle θ is the map

$$\begin{aligned}
R_{l,\theta} = \varphi^{-1} \circ R_{z,\theta} \circ \varphi &= \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta\\ 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta\\ 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta\end{bmatrix}.
\end{aligned}$$

Notice that this matrix fixes all points on the axis

$$l = \langle(1,1,1)\rangle = \{(t,t,t) \in \mathbb{R}^3 : t \in \mathbb{R}\}.$$

(c) The answer is read off from the third column of our final matrix from part (b). The point $(0,0,1)$ is sent to

$$\tfrac{1}{3}(1-\cos\theta+\sqrt{3}\sin\theta, 1-\cos\theta-\sqrt{3}\sin\theta, 1+2\cos\theta).$$

(d) We assume that $\theta \neq 0$ (otherwise $R_{A,\theta} = \text{Id}$ fixes all points). We can think of $R_{A,\theta}$ as a rotation first of $\mathbb{R}^3$, so it fixes only the axis A. This axis intersects S^2 in two antipodal points (by the proof of Problem 4(a) above), so the fixed points of $R_{A,\theta}$ on S^2 are these points.

Here's another proof: the rotation $R_{A,\theta}$ can be decomposed into the composition of two reflections, $R_{A,\theta} = \overline{r}_{M_2} \circ \overline{r}_{M_1}$, where M_1 and M_2 meet at A, and the angle from M_1 counterclockwise (at A) to M_2 is $\theta/2$. From Problem 4(a), we know that $M_2 \cap M_1 = \{P, Q\}$ is exactly two points. These points are certainly fixed by $\overline{r}_{M_2} \circ \overline{r}_{M_1}$, since they lie on both lines M_1 and M_2 (reflection through a line fixes all points on that line).

Finally, we show that all other points of S^2 are not fixed by $R_{A,\theta} = \overline{r}_{M_2} \circ \overline{r}_{M_1}$. Suppose $T \in S^2$ is a fixed point not equal to P or Q. Then

$$T = R_{A,\theta}(T) = \overline{r}_{M_2} \circ \overline{r}_{M_1}(T).$$

Apply $\overline{r}_{M_2}$ to both sides. Since $\overline{r}_{M_2} \circ \overline{r}_{M_2} = \text{Id}$, we have

$$\overline{r}_{M_2}(T) = \overline{r}_{M_1}(T).$$

If T lies in M_2 but not M_1, then we would have

$$T = \overline{r}_{M_2}(T) = \overline{r}_{M_1}(T) \neq T,$$

a contradiction. Similarly if f T lies in M_1 but not M_2. Therefore, assume that T lies in neither M_1 nor M_2.

From Problem 4(c), we have $M_1 = \Pi_{M_1} \cap S^2$ and $M_2 = \Pi_{M_2} \cap S^2$ for planes Π_{M_1} and Π_{M_2} through the origin. Since T is being reflected through each of these planes, the line segment from T to $\overline{r}_{M_1}(T) = \overline{r}_{M_2}(T) \neq T$ must be perpendicular to both Π_{M_1} and Π_{M_2}. Therefore, Π_{M_1} and Π_{M_2} have the same normal vector and both contain the origin, so $\Pi_{M_1} = \Pi_{M_2}$. But this is a contradiction because these are two planes separated by angle $\theta/2 \neq 0$.

We conclude that no such fixed point T exists, so P and Q are the only fixed points of $R_{A,\theta}$.

(e) From Problem 4(b), we know that the points $P, Q \in S^2$ found above in (d) are antipodal, and we showed that these are the fixed points of $R_{A,\theta}$.

Problem 7. (20 pts) **Real-Life Computation**. Consider the longitude φ (azimuth angle) and latitude θ coordinates on Earth. Suppose the surface of the Earth is spherical, its core is at $(0,0,0) \in \mathbb{R}^3$, and the radius of the Earth is $r = 6378$ kilometers.

In this coordinates, the point $(x, y, z) \in \mathbb{R}^3$ corresponds to

$$(x, y, z) = (r\cos\theta\sin\varphi, r\cos\theta\cos\varphi, r\sin\theta),$$

with $\theta \in [-\pi/2, \pi/2], \varphi \in [-\pi, \pi)$, where $\theta = \pi/2$ is the North Pole and $\theta = -\pi/2$ is the South Pole, and $\varphi \in (0, \pi)$ is East of the Greenwich Meridian, and $\varphi \in [-\pi, 0)$ is West of the Greenwich Meridian.

(a) UC Davis is located at $(\theta, \varphi) = (38.5382°N, 121.7617°W)$ and Barcelona (Spain) at $(\theta, \varphi) = (41.3851°N, 2.1734°E)$. Compute approximately the distance d_{S^2} on the surface of Earth from UC Davis to Barcelona.

(b) UC Berkeley is located at $(\theta, \varphi) = (37.8719°N, 122.2585°W)$. Compute the distance d_{S^2} on the surface of Earth from UC Davis to UC Berkeley.

(c) Compare the distances on the surface of Earth with the corresponding distances considered in $\mathbb{R}^3$. In which case is the distance $d_{\mathbb{R}^3}$ closer to the distance d_{S^2} on the surface of Earth ?

Solution.

We need to modify our definition of distance on the sphere somewhat, since our sphere no longer has radius 1. Suppose we have two points $P, Q \in \mathbb{R}^3$ which are both located on the sphere of radius r centered at the origin. Investigating the original derivation for spherical distance in Fig. 3.1 of Stillwell (p. 46), we need the modification

$$d_{S^2}(P, Q) = 2r\arcsin\left(\tfrac{1}{2r} d_{\mathbb{R}^3}(P, Q)\right) = 2r\arcsin\left(\tfrac{1}{2} d_{\mathbb{R}^3}(P/r, Q/r)\right). \tag{2}$$

Note that this reduces to the usual case when $r = 1$. The final formula above says that first we normalize P and Q so that they are on the usual unit sphere, then we find their usual distance in the unit sphere, and then we scale back up. This makes sense, because distances should go up in proportion to the radius in the sphere.

Remark. Also note that it would have been incorrect to simply scale our usual formula by r (meaning $2r\arcsin\left(\tfrac{1}{2} d_{\mathbb{R}^3}(P, Q)\right)$). One way to see that this won't work is that $\tfrac{1}{2} d_{\mathbb{R}^3}(P, Q)$ may not even be in the domain of arcsin if P, Q lie on a large sphere (as they do in this problem).

(a) First we should make our coordinates (θ, φ) match the conventions given in the problem. From the conventions given, North and East are positive, while South and West are negative. Therefore, UC Davis has coordinates

$$(\theta_D, \varphi_D) = (38.5382°, -121.7617°),$$

and Barcelona has coordinates

$$(\theta_B, \varphi_B) = (41.3851°, 2.1734°).$$

On the last page of this document is a Mathematica notebook that calculates distances in $\mathbb{R}^3$ and S^2 for any given attitude and longitude. Play around with it! In short, we calculate the Cartesian coordinates of these points in space, find their distance in $\mathbb{R}^3$, and then put that distance in (2) to find the distance on the Earth. We find that that the distance on Earth from UC Davis to Barcelona is

$$d_{S^2}(D, B) \approx 9480.36 \text{ km}.$$

(b) UC Berkeley has coordinates $(\theta_C, \varphi_C) = (37.8719°, -122.2585°)$. We find that that the distance on Earth from UC Davis to UC Berkeley is

$$d_{S^2}(D, C) \approx 85.9633 \text{ km}.$$

(c) The distances in $\mathbb{R}^3$ are

$$d_{\mathbb{R}^3}(D, B) \approx 8631.39 \text{ km} \qquad \text{and} \qquad d_{\mathbb{R}^3}(D, C) \approx 85.9627 \text{ km}.$$

From UC Davis to UC Berkeley, traveling by foot (on the Earth) rather than by laser (in $\mathbb{R}^3$) increases your trip by less than a meter. But the comparison on a trip to Barcelona increases your trip by nearly 850 kilometers! If you're flying, that should give you enough time to watch about half a movie more than you otherwise would. Certainly the difference is smaller on the shorter trip to Berkeley.

Remark. Notice that the distance on Earth is always larger than the distance in $\mathbb{R}^3$, because arcsin is a convex function on its positive domain. And since the first order Taylor approximation of arcsin (at $x = 0$) is

$$\arcsin x = x + \mathcal{O}(x^3),$$

we see that $d_{\mathbb{R}^3}$ is a very good approximation for d_{S^2} for reasonably small distances (such as a day trip to Berkeley). However, the error becomes larger as x strays too far from 0, explaining why the difference is significant on a trip to Barcelona.

It is also interesting to note that this approximation is better for larger planets. This is easily seen from the Taylor expansion for $2r \arcsin\big(x/2r\big)$ (where r is the radius of the planet), but it can also be viewed geometrically. The surface of a larger planet is more similar to its tangent planes than is the surface of a smaller planet. In other words, larger spheres have smaller *curvature* than smaller spheres. Curvature is measured by second derivatives, and the magnitude of the second derivative is what measures how badly the first order Taylor polynomial approximates a given function.

Click on the link below to see the Mathematica notebook mentioned above. If you'd like to play around with the notebook, click on "Make Your Own Copy" on the upper right. It's free to make a Wolfram ID account and use Mathematica as part of Wolfram Cloud:

https://www.wolframcloud.com/obj/srubin0/Published/Distance_on_Earth.nb

Here is the text of the notebook:

```
r=6378;            (* r is the radius of your "Earth" *)

  (* Put in any coordinates for these pairs to test out other locations. *)
    (* These are the three points from the problem  *)
tD=38.5382;            (* t is the N/S theta coordinate (latitude) in degrees *)
pD=-121.7617;          (* p is the E/W phi coordinate (longitude) in degrees *)

tB=41.3851;
pB=2.1734;

tC=37.8719;
pC=-122.2585;

  (* These calculate Cartesian coordinates given latitude and longitude *)
CartX[t_,p_]:=Cos[t Degree]*Sin[p Degree ];
CartY[t_,p_]:=Cos[t Degree]*Cos[p Degree];
CartZ[t_,p_]:=Sin[t Degree];

  (* Given two coordinate pairs (in latitude and longitude), this calculates *)
    (* the R^3 distance between the two points on the Earth with those coordinates *)
(SpaceDistance[t1_,p1_,t2_,p2_]:=r*EuclideanDistance[
{CartX[t1,p1],CartY[t1,p1],CartZ[t1,p1]},{CartX[t2,p2],CartY[t2,p2],CartZ[t2,p2]}])

  (* This turns the R^3 distance into the distance on the surface of the Earth *)
EarthDistance[t1_,p1_,t2_,p2_]:=Simplify[2*r*ArcSin[(1/(2*r))*SpaceDistance[t1,p1,t2,p2]]]

  (* If you want to test EarthDistance with nice whole numbers for degrees, *)
    (* put it inside a N[] to evaluate numerically. Otherwise, it gives the *)
    (* exact answer in terms of a single degree, o=Pi/180. *)

SpaceDistance[tD,pD,tB,pB]   (* Distance by laser from UC Davis to Barcelona *)
EarthDistance[tD,pD,tB,pB]   (* Distance by foot from UC Davis to Barcelona *)

SpaceDistance[tD,pD,tC,pC]   (* Distance by laser from UC Davis to UC Berkeley *)
EarthDistance[tD,pD,tC,pC]   (* Distance by foot from UC Davis to UC Berkeley *)
```