SOLUTIONS TO PROBLEM SET 5

MAT 141

Abstract. These are the solutions to Problem Set 5 for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. The problems were posted online on Friday Feb 21 and due Friday Feb 28 at 10:00am.

Problem 1. Show that there are *no* parallel lines $L_1, L_2 \subseteq S^2$ in the 2-sphere, i.e. if $L_1, L_2 \subseteq S^2$ are lines, then $L_1 \cap L_2$ is non-empty.

Solution. This is the content of Problem 4(a) in Problem Set 4.

Problem 2. **Triangles in the Euclidean Plane.** Let $T \subseteq \mathbb{R}^2$ be a triangle, and α, β, γ be the interior angles of T.

(a) Show that $\alpha + \beta + \gamma = \pi$.

(b) Construct a triangle T' with the same interior angles α, β, γ, such that

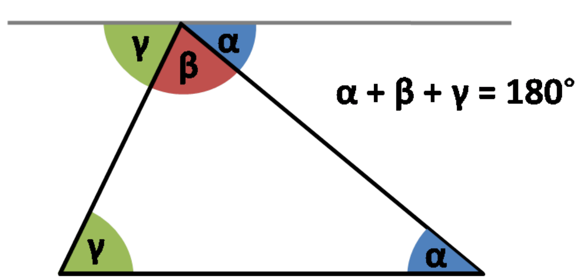
$$Area(T') = 293 \cdot Area(T).$$

(c) Prove that it is not possible to compute the area of a triangle $T \subseteq \mathbb{R}^2$ just by knowing its interior angles α, β, γ.

Solution.

(a) See the figure below. We slide one side of T over and use properties of parallel lines (and interior angles of a transversal) to see that α, β, and γ together fit into an angle of π. Figure from

`https://www.omnicalculator.com/math/triangle-angle`



(b) The dilation $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $\varphi(x, y) = (\sqrt{293}x, \sqrt{293}y)$ preserves angles and has determinant 293. Therefore, $T' := \varphi(T)$ is a triangle with the desired properties.

(c) From part (b), we have two triangles with the same angles but different areas, so the angles of a triangle do not uniquely determine its area. This is to be contrasted with the spherical case, where the area of a triangle with angles α, β, γ is *exactly* equal to $\alpha + \beta + \gamma - \pi$.

Problem 3. (20 pts) **Triangles in Euclidean Surfaces.** In this problem C is the Euclidean cylinder, M is the twisted Euclidean cylinder, T^2 is the Euclidean 2-torus, and K is the Klein bottle. We will denote an arbitrary Euclidean surface by S.

(a) Give two triangles $T_1, T_2 \subseteq C$ such that the interior angles of T_1 coincide with the interior angles of T_2, but the area of T_1 is distinct from the area of T_2.

(b) Do there exist two triangles $T_1, T_2 \subseteq T^2$ with the same interior angles but different area ? How about $T_1, T_2 \subseteq K$ in the Klein bottle ?

(c) Let $T \subseteq S$ be a triangle with sides given by the lines $L_1, L_2, L_3 \subseteq S$. How many regions does the complement $S \setminus \{L_1, L_2, L_3\}$ has ?

Answer for each of the five cases $S = \mathbb{R}^2, S = C, S = M, S = T^2$ and $S = K$.

(d) Let $T \subseteq S$ be a triangle in an arbitrary Euclidean surface S, and α, β, γ be the interior angles of T. Show that $\alpha + \beta + \gamma = \pi$.

Solution.

(a) Every Euclidean surface can be covered with small disks that are isometric to disks in $\mathbb{R}^2$. So as long as our triangles stay within such a local disk, we can mimic geometry in the plane. More simply, if we draw triangles in $\mathbb{R}^2$ that lie completely in a fundamental domain of S, then the images of these triangles under the quotient map $\pi : \mathbb{R}^2 \longrightarrow S$ will satisfy what we need.

In $\mathbb{R}^2$, take the triangle Δ_1 with vertices $(0,0)$, $(1/4,0)$, and $(0,1/4)$. Take another triangle Δ_2 with vertices $(0,0)$, $(1/8,0)$, and $(0,1/8)$. These triangles have the same interior angles but different areas. Let $\pi : \mathbb{R}^2 \longrightarrow C$ be the quotient map to the cylinder, and define $T_1 := \pi(\Delta_1)$ and $T_2 := \pi(\Delta_2)$. Then T_1 and T_2 have the same angles as Δ_1 and Δ_2, because π is a local isometry. Furthermore, π preserves the areas of Δ_1 and Δ_2, because both are contained in a Euclidean disk (a disk of radius $\frac{1}{2}$). We conclude that T_1 and T_2 satisfy the desired properties.

(b) Yes, for both. The argument given in (a) will work for every Euclidean surface.

(c) For $S = \mathbb{R}^2$, a picture easily shows that the complement is split into 7 pieces: T itself, and six unbounded regions.

For $S = C$ and $S = M$, there are infinitely many regions, which you can see, for example, by drawing the lines $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$, and $L_3 = \{y = x - \frac{1}{2}\}$ in a fundamental domain.

For $S = T^2$ and $S = K$, there are many different possible numbers of regions. For example, take the lines $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$, and $L_3 = \{ny = x - \frac{1}{2n}\}$ for any positive integer n (the $-\frac{1}{2n}$ is just there to shift our line a little so we avoid a triple intersection). If you draw these in a fundamental domain for T^2, you can count $n+2$ regions. The same lines in K give $3n+3$ regions.

(d) Let $\pi : \mathbb{R}^2 \longrightarrow S$ be the quotient map. Let L_1, L_2, L_3 be the three non-coincident lines in S that make up T. Then $L_1 = \pi(N_1)$, $L_2 = \pi(N_2)$, and $L_3 = \pi(N_3)$ for some non-coincident lines N_1, N_2, and N_3 in $\mathbb{R}^2$ which form a triangle. Since π is a local isometry, it preserves angles, so the angles formed by N_1, N_2, and N_3 must be the α, β, and γ from $T \subseteq S$. Then $\alpha + \beta + \gamma = \pi$ by Problem 2(a).

Problem 4. (20 pts) **Triangles in S^2 (Part I)**. Let $T \subseteq S^2$ be a triangle in S^2, defined by the lines $L_1, L_2, L_3 \subseteq S^2$.

(a) Show that the area of the unit radius 2-sphere is 4π.

(b) Show that the area of a sector of angle α is $\alpha/2\pi$ the area of the 2-sphere. A *sector* of angle α is the bigon described by two lines at angle α.

(c) Show that the complement $S^2 \setminus \{L_1, L_2, L_3\}$ consists of eight *triangular* regions, where $L_1, L_2, L_3 \subseteq S^2$ are lines defining a triangle.

(d) Show that there exist six pairs of such regions such that the union of the pair of region is a sector as in Part (b).

(e) Let $T \subseteq S^2$ be a triangle, and α, β, γ be the interior angles of T. Show that

$$\alpha + \beta + \gamma = \pi + Area(T).$$

Solution.

(a) Probably the only calculus in the whole course! In the usual spherical coordinates ($\phi \in [0, 2\pi)$ is the azimuthal angle from the x-axis and $\theta \in [0, \pi)$ is the polar angle from the z-axis) an area element on the sphere is $\sin\theta \, d\theta \, d\phi$. We then have

$$\text{Area}(S^2) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi = 4\pi.$$

(b) Let $U \subseteq S^2$ be our sector By a suitable isometry, assume that the vertices of U are the North and South poles, so that the lines creating the sector are lines of longitude. For any curve of constant latitude (the intersection of S^2 with a plane parallel to the xy-axis), its intersection with U forms an arc of angle $\alpha/2\pi$. Build U up as a Riemann sum of horizontal strips by considering n such curves of constant latitude, breaking U into n "rectangular" strips, $R_1, \ldots, R_n$. For large n, each strip R_i has area approximately $\alpha/2\pi$ times the full circular strip containing R_i (this circular strip is the space in S^2 between two nearby horizontal planes). Summing up, the result follows as $n \to \infty$.

(c)-(e) For the rest, see the proof of the Theorem by Harriot in section 3.8 of Stillwell, pp. 65-67.

Problem 5. (20 pts) **Triangles in S^2 (Part II)**.

(a) Show that there exists a triangle $T \subseteq S^2$ such that all its interior angles α, β, γ are distinct from $\pi/2$.

(b) Let $\varepsilon \in \mathbb{R}^+$, prove that there exist triangles $T \subseteq S^2$ such that *one* of its interior angles is less than ε. That is, there are triangles with one of its angles arbitrarily small.

(c) Let $\varepsilon \in \mathbb{R}^+$, prove that there exist triangles $T \subseteq S^2$ such that *two* of its interior angles are less than ε. That is, there are triangles with two of its angles arbitrarily small.

(d) Let $\varepsilon \in \mathbb{R}^+$ be given. Construct a triangle $T \subseteq S^2$ whose area is $2\pi - \varepsilon$, i.e. build a triangle whose area is arbitrarily close to half the area of the 2-sphere S^2.

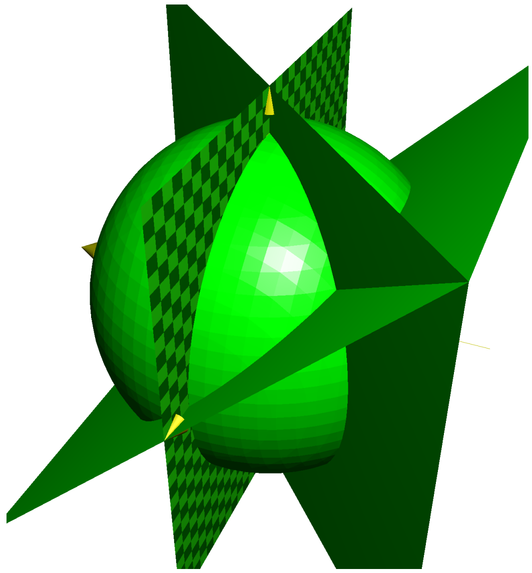
(e) Is is possible to have triangles $T \subseteq S^2$ with arbitrarily small area ?

Solution.

(a) A triangle in S^2 is specified by three non-coincident lines and a selection of one of the eight resulting regions cut out by these lines. The angles in one of these regions differ only by taking supplementary angles of the angles in another region, so we can ignore this selection of region.

Therefore, the problem reduces to showing that there exist three planes in $\mathbb{R}^3$, each passing through the origin, which have no common intersection on S^2, and no two of which are perpendicular. Take the planes $\{x = 0\}$, $\{x + y = 0\}$, and $\{x + z = 0\}$. The intersection of these planes is the single point $(0, 0, 0)$, and their normal vectors are $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 0, 1)$, respectively, no two of which are perpendicular. Therefore, these planes cut out a triangle in S^2 with no right angles.

Shown below is a picture of the above situation. Notice that there are no right angles.



(b) This follows from (c) below.

(c) It suffices to modify the planes in the proof of (a) so that, of the three possible dot products of the resulting normal vectors (which we should now take to be normalized), at least two are arbitrarily close to 1. This will translate directly into arbitrary smallness of the corresponding angles between our planes (since the transition from dot product to angle is arccos, which is one-to-one and takes values near 0 for inputs near 1).

Note that, again, supplementary angles need not concern us: when cutting S^2 with three planes, eight regions are created, grouped into four pairs of congruent triangles. Each of the four types is obtained from a single triangular region T by replacing two of the angles of T by their supplements. Therefore, if our normal vectors have large enough dot products, we are guaranteed that one type of the resulting triangles will include the two corresponding acute angles.

Try these planes out for size:

$$N_1 = \{x = 0\}, \qquad N_2 = \{x + \delta y = 0\}, \qquad \text{and} \qquad N_3 = \{x + \delta z = 0\},$$

where $\delta > 0$ is some small number (really, δ is some monotonically increasing function of the given ε, but the details don't matter). Once again, these planes intersect only at the origin. The corresponding unit normal vectors are

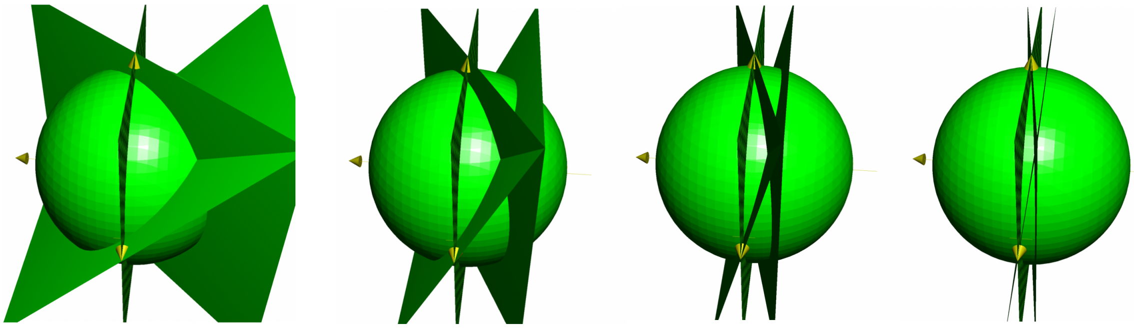
$$\mathbf{n}_1 = (1, 0, 0), \qquad \mathbf{n}_2 = \gamma(1, \delta, 0), \qquad \text{and} \qquad \mathbf{n}_3 = \gamma(1, 0, \delta),$$

where $\gamma = (1 + \delta^2)^{-1/2}$. The pairwise dot products are

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \mathbf{n}_1 \cdot \mathbf{n}_3 = \gamma \qquad \text{and} \qquad \mathbf{n}_2 \cdot \mathbf{n}_3 = \gamma^2.$$

Since $\gamma \to 1$ as $\delta \to 0$, these dot products can be made arbitrarily close to 1. Note that any given triangle cut out by N_1, N_2, and N_3 will not have all of its angles be small (since the angles must add up to something larger than π), but by taking supplements, we can find a triangle where two angle are small.

Shown below are pictures of the situation for values $\delta = 1$, $\delta = 1/2$, $\delta = 1/4$, and $\delta = 1/8$.



(d) Let T be one of the two largest triangles created by the planes N_1, N_2, and N_3 in the solution to (c). Its angles are the supplements of $\arccos\gamma$, $\arccos\gamma$, and $\arccos\gamma^2$, so

$$\begin{aligned}\text{Area}(T) &= (\pi - \arccos\gamma) + (\pi - \arccos\gamma) + (\pi - \arccos\gamma^2) - \pi \\ &= 2\pi - \left(2\arctan\delta + \arccos\left(\frac{1}{1+\delta^2}\right)\right).\end{aligned}$$

Note that the subtracted expression in parentheses is a one-to-one function that goes to 0 as $\delta \to 0$. Therefore, we can set this expression equal to ε and invert, yielding δ as a function of ε. The result is $\text{Area}(T) = 2\pi - \varepsilon$.

(e) Yes. The smallest triangles in our construction above have angles $\arccos\gamma$, $\arccos\gamma$, and $\pi - \arccos\gamma^2$, yielding the area

$$\arccos\gamma + \arccos\gamma + (\pi - \arccos\gamma^2) - \pi = 2\arccos\gamma - \arccos\gamma^2,$$

which goes to 0 as $\delta \to 0$.

Remark. The full range of possible sums of angles for triangles in spherical geometry is $(\pi, 3\pi)$. Contrast to Euclidean geometry where the only allowed value is π.

Problem 6. (20 pts) **Polygons in $\mathbb{R}^2$ and S^2.** Let P be an n-sided polygon, i.e. a region of S^2 bounded by n lines $L_1, L_2, \ldots, L_n$. For instance, $n = 3$ is a triangle.

(a) Let $P \subseteq \mathbb{R}^2$ be a polygon *in the plane*, and $\alpha_1, \ldots, \alpha_n$ its interior angles. Find a formula for the sum $\alpha_1 + \ldots + \alpha_n$ in terms of n.

(b) Find two polygons $P_1, P_2 \subseteq \mathbb{R}^2$ with the same interior angles but different areas.

(c) Let $P \subseteq S^2$ be a polygon *in the sphere*, and $\alpha_1, \ldots, \alpha_n$ its interior angles. Find a formula for the sum $\alpha_1 + \ldots + \alpha_n$ in terms of n and $Area(P)$.

(d) Show that two polygons $P_1, P_2 \subseteq S^2$ with the same interior angles must have the same area.

Solution.

(a) We argue that P can be broken down into $n-2$ triangles without creating any new vertices. This is obvious if P is convex, but otherwise it requires some work. The result $\alpha_1 + \cdots + \alpha_n = (n-2)\pi$ then follows immediately.

We proceed by induction on n. For $n = 3$, the result is obvious. For a general n-gon assume the result for all k-gons with $k < n$. Order the vertices $v_1, \ldots, v_n$ of P in the natural way. Draw a ray from v_1 to v_n and rotate it so that it touches the interior of P. Stop when the ray hits a vertex v_ℓ. If $\ell = n$, start over, and rotate in the other direction. Now we may assume that $i \neq 2, n$. Then $Q_1 := v_1 v_2 \cdots v_\ell$ and $Q_2 = v_1 v_\ell v_{\ell+1} \cdots v_n$ are two polygons that together make up P. Both Q_1 is an ℓ-gon, and Q_2 is an $(n-\ell+2)$-gon.

Notice that ℓ and $n-\ell+2$ are less than n. By the inductive hypothesis, Q_1 can be broken into $\ell-2$ triangles, and Q_2 can be broken into $n-\ell$ triangles. Therefore, P can be broken into $(\ell-2)+(n-\ell) = n-2$ triangles.

(b) Take triangles T and T' from Problem 2(b). In general, a dilation preserves angels of any polygon, but changes the area.

(c) The argument is the same as in (a). We cover P with $n-2$ triangles, whose interior angles sum to $\alpha_1 + \cdots + \alpha_n$. Furthermore, the sum of the areas of these triangles is Area(P), so

$$\text{Area}(P) = \alpha_1 + \cdots + \alpha_n - (n-2)\pi.$$

As usual, the excess in Euclidean geometry is identically zero, while in spherical geometry, it is Area(P).

(d) Since our formula for area takes as input *only* the interior angles, the area of a polygon in the sphere is completely determined by its interior angles.

Problem 7. (20 pts) **Tilings of the 2-sphere S^2**. In this problem we will study how to regularly tile the 2-sphere, i.e. how to regularly cover the 2-sphere with polygons of the same area.

(a) Show that S^2 can be subdivided into four triangles of the same area.

(b) Show that S^2 can be subdivided into eight triangles of the same area.

(c) Show that S^2 can be subdivided into twenty triangles of the same area.

(d) Show that S^2 can be subdivided into six squares of the same area.

(e) Show that S^2 can be subdivided into twelve pentagons of the same area.

(f) (**Challenging**) Show that if S^2 is subdivided into n-gons, all with the same area and same length of sides, then the subdivision must be as into triangles, squares or pentagons.

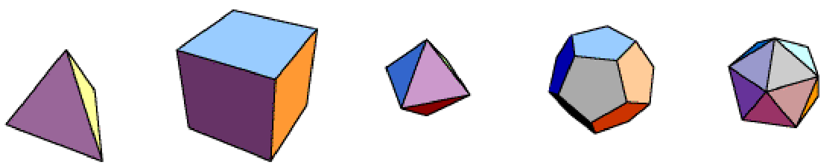
(Part (f) will not be graded.)

Solution.

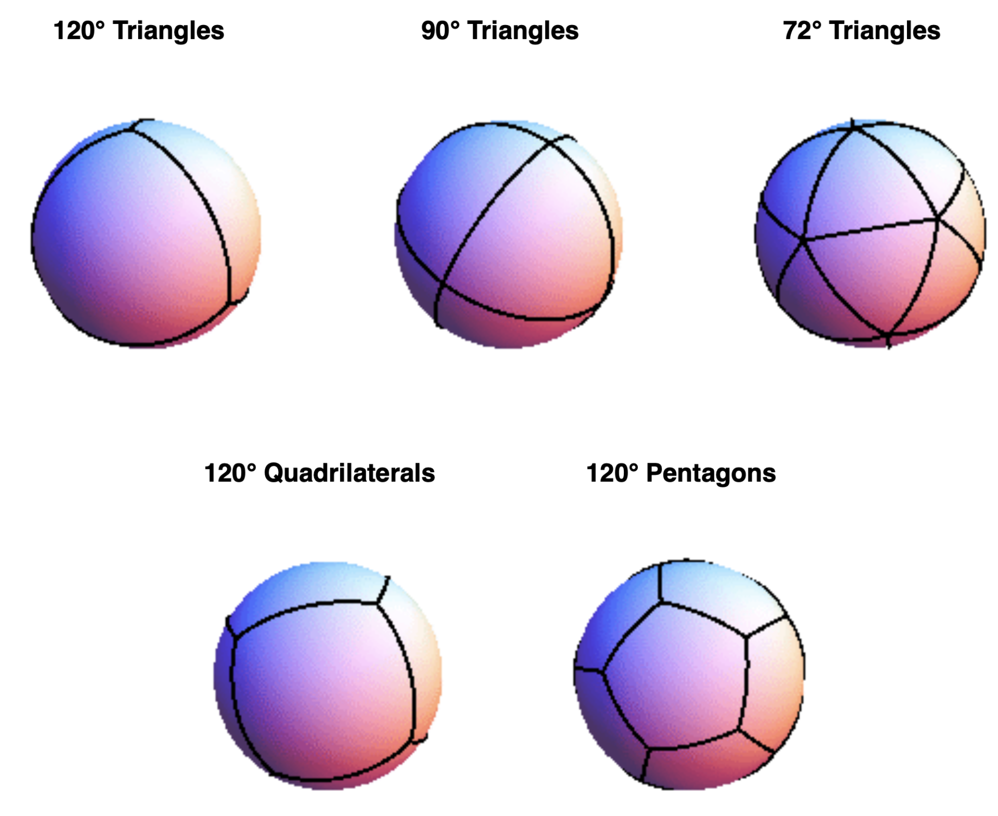
(a)-(e) All images are from

`https://mathstat.slu.edu/escher/index.php/Spherical_Geometry`

Notice that the types and number of polygons in parts (a) through (e) correspond exactly to the five platonic solids, the tetrahedron (four triangles), cube (six squares), octahedron (eight triangles), dodecahedron (twelve pentagons), and icosahedron (twenty triangles), shown below.



Each of these regular polyhedra may be used to tile the sphere as instructed, simply by putting the vertices of a platonic solid on the sphere, erasing the original faces, and replacing the edges with great circles. In a way, the faces are “ballooned” up to the sphere. The results look like this:



(f) This is part of the classification of regular polyhedra: the faces of any regular polyhedron are triangles, squares or pentagons. At any given vertex of our spherical tiling, the sum of the angles at that vertex is 2π. Since there must be at least three edges coming from each vertex, this means that all polygons used must have interior angles at most $2\pi/3$. However, from Problem 6(c), an interior angle θ of a regular spherical n-gon is strictly greater than $(1-2/n)\pi$. Therefore, we have

$$(1-2/n)\pi < \theta \leq 2\pi/3,$$

which implies $n < 6$. Therefore, the only possibilities are $n = 3, 4, 5$, corresponding to triangles, squares, or pentagons. In particular, for $n = 5$, $\theta > 0.6\pi$, so there can only be 3 pentagrams meeting at a vertex (any more would exceed a total angles of 2π). This is the dodecahedron of part (e). Similarly, for $n = 4$, $\theta > \pi/2$, so only 3 squares may meet at a vertex, the cube of part (d). Finally, for $n = 3$, $\theta > \pi/3$, and we can have 3, 4, or 5 triangles meeting at a vertex, corresponding to the tetrahedron, octahedron, and icosahedron of parts (a), (b), and (c), respectively.

Here is another proof: Suppose that we have partitioned S^2 into n-gons, and let q be the number of edges that meet at every vertex. Let V, E, and F stand for the number of vertices, edges, and polygons used to partition the sphere. We will make use of Euler's formula (without proof), which says that $V - E + F = 2$.

First, notice that since every polygon has n edges, we might say that there are nF total edges. This double counts edges, since every edge belongs to two faces. Therefore, $nF = 2E$ actually. Similarly, since q edges meet at each vertex, we might say that there are qV edges. This also double costs edges, because each edge has two vertices in it, so $qV = 2E$ actually. Plugging into Euler's formula and simplifying yields

$$\frac{1}{n} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.$$

First, we automatically have $n \geq 3$ (because there are no polygons with less than three sides), and $q \geq 3$ (a polygon must have at least three vertices). Next, we can't have both n and q greater than 3, because otherwise we would have

$$\frac{1}{2} + \frac{1}{E} = \frac{1}{n} + \frac{1}{q} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

a contradiction because $E > 0$. Therefore, we either have $n = 3$ or $q = 3$. If $n = 3$, then we are in the case of a partition by triangles. If $q = 3$, then we can simplify Euler's formula to

$$\frac{1}{n} - \frac{1}{6} = \frac{1}{E}.$$

The only solutions for n are $n = 3$, $n = 4$, or $n = 5$, corresponding to subdivisions of the sphere by triangles, squares, or pentagons, as desired.

We could keep going and use Euler's formula to compute all possible combinations of the numbers (n, q, V, E, F). These cover the cases of parts (a) through (e).

Remark. Actually, if we allow for 2-gons, we have additional spherical tessellations, like the one below.