SOLUTIONS TO PROBLEM SET 6

MAT 141

Abstract. These are the solutions to Problem Set 6 for the Euclidean and Non-Euclidean Geometry Course in the Winter Quarter 2020. The problems were posted online on Saturday Feb 29 and due Friday March 6 at 10:00am.

Problem 1. (20 pts) **Regions in Euclidean Plane and 2-Sphere.** A set of lines $\mathcal{L} := \{L_1, L_2, \ldots, L_n\}$ be a set of n lines is said to be *generic* if not two lines in $\mathcal{L}$ are parallel, and no three lines in $\mathcal{L}$ intersect at a point.

(a) Let $\mathcal{L} = \{L_1, L_2, \ldots, L_n\}$ be a generic set of n lines in $\mathbb{R}^2$. Show that the complement
$$\mathbb{R}^2 \setminus (L_1 \cup L_2 \cup \ldots \cup L_n),$$
of the lines in $\mathcal{L}$ inside the Euclidean plane $\mathbb{R}^2$ consists of $\dfrac{n^2+n+2}{2}$ regions.

(b) Show that Part.(a) might *not* hold if $\mathcal{L}$ is not necessarily a generic set.

(c) Let $\mathcal{L} = \{L_1, L_2, \ldots, L_n\}$ be a generic set of n lines in S^2. Show that the complement
$$S^2 \setminus (L_1 \cup L_2 \cup \ldots \cup L_n),$$
of the lines in $\mathcal{L}$ inside the Euclidean plane S^2 consists of $n^2 - n + 2$ regions.

(d) Show that Part.(c) might *not* hold if $\mathcal{L}$ is not necessarily a generic set.

(e) Consider $\mathcal{L}$ as in Part.(c). Show that the *total* number of intersections of all the lines in $\mathcal{L}$ is $n(n-1)$.

Solution.

(a) We prove this formula by induction on the number n of lines in $\mathcal{L}$. It is clearly true for $n = 0$. Now take n to be some general number of lines, and assume the formula holds for the first $n-1$ lines. That is, we are assuming that
$$\mathbb{R}^2 \setminus \{L_1 \cup L_2 \cup \cdots \cup L_{n-1}\}$$
consists of
$$\frac{(n-1)^2 + (n-1) + 2}{2}$$
regions. Since no two lines are parallel, the final line L_n crosses through all $n-1$ previous lines, introducing a new region with each new line that it cuts. Therefore, cutting with L_n introduces n new regions (there is an extra region

created after crossing the final line as well), for a grand total of

$$\frac{(n-1)^2+(n-1)+2}{2}+n=\frac{n^2+n+2}{2}$$

regions.

(b) Take any $n \geq 1$ lines that pass through a common point. The complement of these lines in $\mathbb{R}^2$ is $2n$ regions, which is distinct from $(n^2+n+2)/2$ if $n \neq 1, 2$. Or, take n parallel lines, for a total of $n+1$ regions, which is distinct from $(n^2+n+2)/2$ if $n \neq 1$. Of course, for $n = 1$, all lines are generic.

(c) We again use induction on the number n of lines in $\mathcal{L}$. The formula is clearly true for $n = 1$. Now take n to be some general number of lines, and assume the formula holds for the first $n-1$ lines. That is, we are assuming that

$$S^2 \setminus \{L_1 \cup L_2 \cup \cdots \cup L_{n-1}\}$$

consists of

$$(n-1)^2-(n-1)+2$$

regions. The last great circle L_n crosses each of the other $n-1$ great circles twice, introducing $2(n-1)$ new regions. Therefore, the total number of regions is now

$$(n-1)^2-(n-1)+2+2(n-1)=n^2-n+2.$$

(d) Again take any n lines that pass through a common point. The complement in S^2 is again $2n$ regions, which is distinct from n^2-n+2 if $n \neq 1, 2$. Any set of 1 or 2 lines is trivially generic.

(e) Each pair of lines in $\mathcal{L}$ intersects in two antipodal points. Since no two different pairs share any of these points, the total number of intersections is twice the number of pairs. That is, there are

$$2 \cdot \binom{n}{2} = n(n-1)$$

total intersections.

Problem 2. (20 pts) **Stereographic Projection From North and South Poles.** Let $N = (0,0,1) \in S^2$ be the North pole and $S = (0,0,-1)$ the South pole of the 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$.

Let us identify the 2-plane $\Pi = \{(x,y,z) \in \mathbb{R}^3 : z = 0\} \subseteq \mathbb{R}^3$ with the Euclidean plane $\mathbb{R}^2 = \{(u,v) : u, v \in \mathbb{R}^2\}$ via $(x,y,0) = (u,v)$.

(a) Let $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ be the stereographic projection from the North pole N, defined by

$$\pi_N(P) = Q, \quad P \in S^2, Q \in \mathbb{R}^2,$$

where Q is the unique intersection point $L_{P,N} \cap \Pi$ distinct from N, and $L_{P,N} \subseteq \mathbb{R}^3$ is the unique line containing $P, N \in \mathbb{R}^3$. Find a formula of $\pi_N(P) = (u,v)$

in terms of $P = (x, y, z)$.

(b) Find a formula for the inverse map $\pi_N^{-1} : \mathbb{R}^2 \longrightarrow S^2 \setminus \{N\}$, i.e. the unique map such that $\pi_N^{-1} \circ \pi_N = id_{S^2}$ and $\pi_N \circ \pi_N^{-1} = id_{\mathbb{R}^2}$.

(c) Let $\pi_S : S^2 \setminus \{S\} \longrightarrow \mathbb{R}^2$ be the stereographic projection from the South pole S, defined by

$$\pi_S(P) = Q, \quad P \in S^2, Q \in \mathbb{R}^2,$$

where Q is the unique intersection point $L_{P,S} \cap \Pi$ distinct from S, and $L_{P,S} \subseteq \mathbb{R}^3$ is the unique line containing $P, S \in \mathbb{R}^3$. Find a formula of $\pi_S(P) = (u, v)$ in terms of $P = (x, y, z)$.

(d) Show that there exists an isometry $\varphi : S^2 \longrightarrow S^2$ such that $\pi_S = \pi_N \circ \varphi$. Is this isometry unique ?

Solution.

(a) Let's give our point coordinates: $P = (x, y, z) \in S^2 \subseteq \mathbb{R}^3$. The line $L_{P,N}$ is best written out with a parametrization by starting at N and using the velocity vector $P - N = (x, y, z - 1)$. Then

$$L_{P,N} = \{N + t(P - N) : t \in \mathbb{R}\} = \{(tx, ty, t(z - 1) + 1) : t \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

This crosses the plane Π when $t(z - 1) + 1 = 0$, or $t = 1/(1 - z)$. The full coordinates for this value of t are $(x/(1 - z), y/(1 - z), 0)$, so we see that

$$(u, v) = \pi_N(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right). \tag{1}$$

(b) We'll play the same game as in (a). For a point $U \in \Pi$ in the plane, with coordinates $U = (u, v, 0)$, the line from the North pole N to U is

$$L_{U,N} = \{(tu, tv, -t + 1) : t \in \mathbb{R}\}.$$

We want to know where this line crosses the sphere, so we require

$$(tu)^2 + (tv)^2 + (-t + 1)^2 = 1.$$

Throwing out the solution $t = 0$ (this corresponds to the North pole), we find

$$t = \frac{2}{u^2 + v^2 + 1}.$$

Substituting back into the line $L_{U,N}$ and using

$$-\frac{2}{u^2 + v^2 + 1} + 1 = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1},$$

we find

$$(x, y, z) = \pi_N^{-1}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).$$

The line $L_{U,N}$ connecting U to N and intersecting P in the sphere S^2 is exactly the line $L_{P,N}$ connecting P to N and intersecting U in the plane Π. This confirms that π_N^{-1} is a proper inverse on both sides of π_N.

(c) We repeat the process from before. Now, for $P = (x, y, z)$, the velocity vector is $P - S = (x, y, z + 1)$, and

$$L_{P,S} = \{S + t(P - S) : t \in \mathbb{R}\} = \{(tx, ty, t(z + 1) - 1) : t \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

The third coordinate is 0 when $t = 1/(1 + z)$, so

$$(u, v) = \pi_S(x, y, z) = \left(\frac{x}{1 + z}, \frac{y}{1 + z}\right). \tag{2}$$

(d) Let $\varphi : S^2 \longrightarrow S^2$ be reflection in the xy-plane Π. That is,

$$\varphi(x, y, z) = \overline{r}_E(x, y, z) = (x, y, -z).$$

Then

$$\pi_N \circ \varphi(x, y, z) = \pi_N(x, y, -z) = \left(\frac{x}{1 + z}, \frac{y}{1 + z}\right) = \pi_S(x, y, z).$$

as desired.

The isometry φ is indeed unique. In fact, it is the only set-wise map, isometric or not, that can do this. In trying to solve the formula $\pi_S = \pi_N \circ \varphi$ for φ, we could just compose both sides on the left with π_N^{-1} to uniquely pin down $\pi_N^{-1} \circ \pi_S = \varphi$. Since the left-hand side is a precise function, φ has no choice but to match it. We could have calculated φ in this way, but it was easier to just guess that reflection would work.

Remark. It was somewhat special that $\pi_N^{-1} \circ \pi_S$ turned out to be an isometry at all, given that it is the composition of two *very* non-isometric projections. This result heuristically tells us that these two projections are essentially the same, since they "agree up to isometry".

Problem 3. **Properties of π_N.** Let $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ be the stereographic projection from the North pole.

(a) Show that π_N is bijective.

(b) Let $P \in S^2$ a point in the equator $E = S^2 \cap \Pi$. Show that $\pi_N(x, y, z) = (x, y)$.

(c) Describe the image of the lower hemisphere $S^2 \cap \{(x, y, z) : z \leq 0\}$.

(d) Describe the image of the upper hemisphere $(S^2 \setminus \{N\}) \cap \{(x, y, z) : z \geq 0\}$.

(e) Let $\theta \in S^1$ be an angle. Find an isometry $\phi \in \mathrm{Iso}(S^2 \setminus \{N\})$ such that

$$\pi_N \circ \phi = R_{(0,0),\theta} \circ \pi_N,$$

where $R_{(0,0),\theta}$ in the rotation of angle θ centered at the origin $(0, 0) \in \mathbb{R}^2_{u,v}$.

Solution.

(a) A map is bijective if and only it has a two-sided inverse. We constructed such an inverse for π_N in Problem 1(b), so stereographic projection is definitely bijective. The take away from this is that we get a natural correspondence between the sphere (minus a point) and the plane.

(b) This is true geometrically because the line $L_{P,N}$ from P to the North pole crosses the plane Π already at P. Symbolically, since P is on the equator, it has the form $P = (x, y, 0)$. Then

$$\pi_N(P) = \pi_N(x, y, 0) = \left(\frac{x}{1-0}, \frac{y}{1-0}\right) = (x, y).$$

(c) We will show that the lower hemisphere is mapped under π_N to the unit disk in the plane,

$$D = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1.$$

The image of $P = (x, y, z)$ under π_N is $\frac{1}{1-z}(x, y)$, which has norm

$$\frac{x^2 + y^2}{(z-1)^2} = \frac{\sqrt{1-z^2}}{(z-1)^2} = \sqrt{\frac{1-z^2}{(z-1)^4}}.$$

We will show that the final expression under the square root is at most 1 when $z \leq 0$. Call this expression $r(z)$. Clearly $r(0) = 1$. Furthermore $r'(z) = 2(z+2)/(z-1)^4$, which is positive for all $z < 1$ (and all points in $S^2 \setminus \{N\}$ have z-coordinate less than 1). Therefore $r(z) \leq 1$ for $z < 0$.

This proves that $\pi_N(P)$ is in the unit disk D. We also want to show that the *entire* disk is actually covered under π_N by the lower hemisphere. Indeed, a point in D has the form (u, v) for $u^2 + v^2 \leq 1$. Under π_N^{-1}, this corresponds in the sphere to

$$\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right). \tag{3}$$

Since $u^2 + v^2 - 1 \leq 0$, we see that the third coordinate of the image is negative, so our point (u, v) is the image under π_N of some point in S^2 located in the lower hemisphere.

(d) Since π_N is a bijection which maps the lower hemisphere to the unit disk D, it must map the upper hemisphere to the remaining part of the plane, $\mathbb{R}^2 \setminus D$ (everything *but* the unit disk).

More directly, the fact that $r'(z)$ above is positive for all $z < 1$ gives us immediately that $r(z) \geq 1$ for all $0 \leq z < 1$, which takes care of the whole upper hemisphere. The opposite direction is also similar to the lower hemisphere. Take a point (u, v) in $\mathbb{R}^2 \setminus D$, meaning $u^2 + v^2 \geq 1$. Now $u^2 + v^2 - 1 \geq 0$, so the third coordinate in 3 is positive. Therefore, (u, v) is the image under π_N of some point in the upper hemisphere of S^2.

Remark. We now see that the correspondence induced by stereographic projection can be refined to three separate correspondences: the equator with itself, the lower hemisphere with the unit disk, and the upper hemisphere with the complement of the unit disk.

(e) Stereographic projection is a phenomenon mostly about the z-axis, and it is totally uniform under rotations about this axis. Therefore, our guess should be that ϕ is essentially the same map as $R_{(0,0),\theta}$, a rotation about the z axis by θ. Our notation was $R_{z,\theta}$, which was really just the matrix $R_{(0,0),\theta}$ in the xy-plane plus no movement in the z direction. Formally,

$$\begin{aligned}
\pi_N \circ R_{z,\theta}(x, y, z) &= \pi_N \left(\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \\
&= \pi_N(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z) \\
&= \frac{1}{1-z}(x\cos\theta - y\sin\theta, x\sin\theta + y) \\
&= \frac{1}{1-z} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= R_{(0,0),\theta} \circ \pi_N(x, y, z),
\end{aligned}$$

so the choice $\phi = R_{z,\theta}$ works.

Remark. If we think of the plane $\mathbb{R}^2_{u,v}$ as just the xy-plane Π in $\mathbb{R}^3$ (which is always fine), then the 2×2 matrix $R_{(0,0),\theta}$ naturally becomes *exactly* the 3×3 matrix $R_{z,\theta}$. So stereographic projection π_N commutes with rotation about the z-axis, as long as we think about things correctly. We can now feel free to perform $R_{z,\theta}$ rotations at will when dealing with stereographic projections. These are our "suitable isometries" that can simplify problems.

Problem 4. (20 pts) **Stereographic Projection and Lines.** Let us consider the stereographic projection $\pi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ from the North pole.

(a) Let $L \subseteq S^2$ be a line. Show that the image $\pi_N(L)$ is a line if and only if L passes through the North pole.

(b) Let $L \subseteq S^2$ be a line such that $N \notin L$. Show that $\pi_N(L)$ is a circle.

(c) Show that π_N is *not* an isometry.

(d) Prove that π_N preserves angles.

Solution.

(a) Technically we should be omitting the north pole $N = (0, 0, 1)$ from all great circles before we plug them into π_N. This will always be assumed, so in practice we restrict to $-1 \leq z < 1$ for the third-coordinate in S^2.

From Problem 4 of Problem Set 5, The line L must necessarily pass through the equator in two antipodal points. By a suitable rotation about the North pole (see the Remark following the solution of Problem 3(e) above), we may assume that L passes through the points $(1, 0, 0)$ and $(-1, 0, 0)$.

If L passes through the North pole, then L is the unit circle in the xz-plane,

$$L = \{(x, 0, z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}.$$

Therefore,

$$\pi_N(L) = \{\pi_N(x, 0, z) : x^2 + z^2 = 1\} = \left\{\left(\frac{x}{1-z}, 0\right) \in \mathbb{R}^2_{u,v} : x^2 + z^2 = 1\right\}.$$

To show that this set is actually the whole u-axis (first coordinate) in $\mathbb{R}^2$, we just need to argue that the equations $x/(1-z) = t$ and $x^2 + z^2 = 1$ can be solved simultaneously for any $t \in \mathbb{R}$. It is readily seen that

$$x = \frac{2t}{t^2+1} \qquad \text{and} \qquad z = \frac{t^2-1}{t^2+1}$$

are the solutions, so we conclude that

$$\pi_N(L) = \{(t, 0) \in \mathbb{R}^2_{u,v} : t \in \mathbb{R}\}$$

is a line in the plane.

Now suppose L does not contain the North pole. By (b) below, $\pi_N(L)$ is a circle, so it cannot be a line in $\mathbb{R}^2$! As a more direct proof, we will show that $\pi_N(L)$ is a *bounded* set in the plane. Since lines in the plane are unbounded, this means that $\pi_N(L)$ cannot be a line.

Since $N \neq L$, L is a full circle in the domain $S^2 \setminus \{N\}$ of π_N. Therefore, there is a point of L, lying in this domain, which achieves the maximum height of L; that is, a point $(\hat{x}, \hat{y}, \hat{z}) \in L$ such that no other point in L has z-coordinate greater than $\hat{z}$. Take any point $(x, y, z) \in L$. We necessarily have $z \leq \hat{z}$, and stereographic projection gives us

$$\pi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

in the plane. Both of the coordinates $x/(1-z)$ and $y/(1-z)$ have magnitude less than or equal to $1/(1-\hat{z})$. Since this applies to any point in L, we see that $\pi_N(L)$ is contained in the square centered at $(0, 0)$ with side length $2/(1-\hat{z})$. Therefore, $\pi_N(L)$ is a bounded set in the plane, so it cannot be a line.

Remark. Note that the argument above fails if L contains the North pole, because then there would *not* be a highest point (we must omit the North pole N from L). This is why $\pi_N(L)$ is an unbounded line if L contains N. The higher you move up on the sphere, the farther from the origin you'll be sent by π_N. As you approach the North pole, your image under π_N approaches "infinity" in the plane.

(b) Stillwell uses complex coordinates in Section 3.4 to show this result by relating reflections in the sphere to inversions in circles in the plane. We present here a more elementary (though perhaps messier) proof using parametrizations.

Since L is a line in S^2 avoiding the North pole, we may perform a suitable R_z isometry and assume that L meets the equator at exactly $(1,0,0)$ and $(-1,0,0)$ (we can ignore the case where L is the equator itself, since the result is obviously true then). Then L is actually the image of the equator under some $R_{x,\theta}$ rotation. By symmetry and a suitable R_z rotation, it suffices to restrict to the case $0<\theta<\pi/2$. A good parametrization of the equator is $(\cos t, \sin t)$, $0 \le t < 2\pi$, so our line L has the parametrization

$$\begin{aligned} L &= \{R_{x,\theta}(\cos t, \sin t) : t \in [0,2\pi)\} \\ &= \{(\cos t, \cos\theta\sin t, \sin\theta\sin t) : t \in t \in [0,2\pi)\}. \end{aligned}$$

Now we stereographically project:

$$\begin{aligned} \pi_N(L) &= \{\pi_N(\cos t, \cos\theta\sin t, \sin\theta\sin t) : t \in t \in [0,2\pi)\} \\ &= \left\{\left(\frac{\cos t}{1-\sin\theta\sin t}, \frac{\cos\theta\sin t}{1-\sin\theta\sin t}\right) : t\in[0,2\pi)\right\}. \end{aligned} \tag{4}$$

We need to argue that this image set is actually the parametrization of some circle in the plane. The problem is that this will not be a parametrization with constant velocity (since π_N is not an isometry).

We first find a candidate for the center and radius of this circle. By the symmetry of the problem, the center of this circle should lie somewhere on the y-axis of the plane. Therefore, we just need to find the extreme y-coordinates of $\pi(L)$. Their average will be the center of our circle, and their common distance to that center will be our radius. The extreme values of

$$\frac{\cos\theta\sin t}{1-\sin\theta\sin t}$$

occur at $t=\pi/2$ and $t=3\pi/2$ which can be verified with calculus. (We could also know that these are correct by noticing that these values correspond to the high and low points of the original circle great circle L. We already know that $\pi_N(P)$ is closer to the origin the lower P is in $\mathbb{R}^3$, and that π_N preserves azimuthal angles. So we can conclude that these values of t will attain the extreme y-values of $\pi_N(L)$.) Substituting in and simplifying, we see that the extreme y-values of $\pi(L)$ are

$$\left(0, \frac{\sin\theta+1}{\cos\theta}\right), \qquad \text{and} \qquad \left(0, \frac{\sin\theta-1}{\cos\theta}\right).$$

Since the x-coordinates of these points are 0, this confirms our intuition that the center of our circle lies on the y-axis.

The average of these extreme y-values is $\tan\theta$, from which each value differs by $\sec\theta$. Therefore, our candidate for $\pi_N(L)$ is the circle

$$C = \{(\sec\theta\cos t, \sec\theta\sin t + \tan\theta) : t \in [0,2\pi)\}.$$

To prove that $\pi_N(L) = C$ we could find a reparametrization for π_N (perhaps by finding the arc-length parameter with integration), but there is an easier clever way. After some trigonometry, you can confirm that the coordinates given for $\pi_N(L)$ in 4 satisfy the equation $x^2 + (y - \tan\theta)^2 = \sec^2\theta$, which proves that $\pi_N(L)$ lies in C. To see that it covers all of C, notice that we already know that $\pi_N(L)$ covers the extreme y-values of C. Since π_N is continuous, the intermediate value theorem tells us that $\pi_N(L)$ must hit all y-values between these extremes. By obvious symmetry, this covering must occur on both paths between extreme y-values, which covers all of C.

(c) There are many ways to see this. In particular, since π_N carries great circles through N to lines in $\mathbb{R}^2$, it takes something with length 2π to something with infinite length. Since isometries preserve distance, stereographic projection is not an isometry.

(d) All angles in S^2 are achieved by the intersection of lines. The angle at the intersection of two lines can be defined as the angle between the planes in $\mathbb{R}^3$ that cut out those lines in S^2, or it can be defined more intrinsically as the angle between the tangent vectors of those lines at the point of intersection. This second definition is very useful in describing isometries, and also in describing the weaker notion of angle-preserving maps (also called *conformal* maps).

Suppose we have two tangent vectors $\mathbf{u}$ and $\mathbf{v}$ on the sphere, both based at the same point $P = (x, y, z) \in S^2$. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}.$$

We want to show that as the map π_N acts on these vectors, it does not change this angle. The way that the function π_N acts on vectors (based at (x, y, z)) is via its derivative matrix (also called the Jacobian),

$$D\pi_N = \begin{bmatrix} \frac{1}{1-z} & 0 & \frac{x}{(1-z)^2} \\ 0 & \frac{1}{1-z} & \frac{y}{(1-z)^2} \end{bmatrix},$$

which we obtained by taking the partial derivatives of 2. This matrix takes in the vectors $\mathbf{u}$ and $\mathbf{v}$ based at P (which have 3 components), and outputs vectors (with 2 components) based at $\pi_N(P)$. To show that the angle $\cos\theta$ is unchanged, we want to prove that

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{D\pi_N\mathbf{u} \cdot D\pi_N\mathbf{v}}{\|D\pi_N\mathbf{u}\| \cdot \|D\pi_N\mathbf{v}\|}. \tag{5}$$

Note that the condition for an isometry would be $\mathbf{u} \cdot \mathbf{v} = D\pi_N\mathbf{u} \cdot D\pi_N\mathbf{v}$, which is certainly not true, because for $\mathbf{u} = \mathbf{v}$, this would say that π_N preserves all norms (which we are about to show is false). Give our vectors the components

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

We will need the following important observation: the tangent plane to S^2 at the point $P = (x, y, z)$ is comprised of the vectors (based at P) which are perpendicular to the radial line from the origin to P. Therefore, we have the relations

$$\begin{aligned} 0 &= P \cdot \mathbf{u} = xu_x + yu_u + zu_z \quad \text{and} \\ 0 &= P \cdot \mathbf{v} = xv_x + yv_u + zv_z. \end{aligned} \tag{6}$$

All that's left to do is calculate. To start,

$$D\pi_N \mathbf{u} = \begin{bmatrix} \frac{1}{1-z} & 0 & \frac{x}{(1-z)^2} \\ 0 & \frac{1}{1-z} & \frac{y}{(1-z)^2} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \frac{1}{1-z} \begin{bmatrix} u_x + \frac{x}{1-z} u_z \\ u_y + \frac{y}{1-z} u_z \end{bmatrix},$$

and similarly for $D\pi_N \mathbf{v}$. Using 6 and $x^2 + y^2 + z^2 = 1$, the norm is

$$\begin{aligned} \|D\pi_N \mathbf{u}\| &= \frac{1}{1-z} \left(\left(u_x + \frac{x}{1-z} u_z \right)^2 + \left(u_y + \frac{y}{1-z} u_z \right)^2 \right)^{1/2} \\ &= \frac{1}{1-z} \left(u_x^2 + u_y^2 + \frac{x^2 + y^2}{(1-z)^2} u_z^2 + \frac{2u_z}{1-z} (xu_x + yu_y) \right)^{1/2} \\ &= \frac{1}{1-z} \left(u_x^2 + u_y^2 + \frac{1 - z^2}{(1-z)^2} u_z^2 + \frac{2u_z}{1-z} (-zu_z) \right)^{1/2} \\ &= \frac{1}{1-z} \left(u_x^2 + u_y^2 + \frac{1+z}{1-z} u_z^2 - \frac{2z}{1-z} u_z^2 \right)^{1/2} \\ &= \frac{1}{1-z} (u_x^2 + u_y^2 + u_z^2)^{1/2} \\ &= \frac{1}{1-z} \|\mathbf{u}\|, \end{aligned}$$

and similarly for $\mathbf{v}$ (note that the norm depends only on the magnitude of our vector and the location of its base, not on its direction).

Substituting this into 5, we now just need to show that

$$(1-z)^2 (D\pi_N \mathbf{u} \cdot D\pi_N \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}.$$

Indeed,

$$\begin{aligned}
(1-z)^2(D\pi_N\mathbf{u}\cdot D\pi_N\mathbf{v}) &= \begin{bmatrix} u_x + \frac{x}{1-z}u_z \\ u_y + \frac{y}{1-z}u_z \end{bmatrix} \cdot \begin{bmatrix} v_x + \frac{x}{1-z}v_z \\ v_y + \frac{y}{1-z}v_z \end{bmatrix} \\
&= u_xv_x + u_yv_y + \frac{x^2+y^2}{(1-z)^2}u_zv_z \\
&\qquad + \frac{1}{1-z}\big(x(u_xv_z+v_xu_z) + y(u_yv_z+v_yu_z)\big) \\
&= u_xv_x + u_yv_y + \frac{1-z^2}{(1-z)^2}u_zv_z \\
&\qquad + \frac{1}{1-z}\big((xu_x+yu_y)v_z + (xv_x+yv_y)u_z\big) \\
&= u_xv_x + u_yv_y + \frac{1+z}{1-z}u_zv_z \\
&\qquad + \frac{1}{1-z}\big((-zu_z)v_z + (-zv_z)u_z\big) \\
&= u_xv_x + u_yv_y + u_zv_z \\
&= \mathbf{u}\cdot\mathbf{v},
\end{aligned}$$

which completes the proof.

If you don't like tangent vectors, here is another more elementary approach, though it is again messier than the approach using complex coordinates. Angles in S^2 are created when lines meet, so we need to look at two intersecting lines. We omit the case where two lines L_1 and L_2 intersect at the North pole, because that's not in the domain of π_N. By a suitable R_z rotation, we can assume that L_1 intersects the equator E at the standard points $(1,0,0)$ and $(-1,0,0)$.

First we consider the case where L_2 also intersects E at $(\pm1,0,0)$. From the construction in the solution to (b) above, we are then in the situation where L_1 is some R_{x,θ_1} rotation of E, and L_2 is some R_{x,θ_2} rotation of E. Clearly the angle between L_1 and L_2 is $\theta_2-\theta_1$, where we can without loss of generality assume that $\theta_2>\theta_1$. Since $\theta_2-\theta_1=(\theta_2-0)-(\theta_1-0)$, we actually only have to prove the case where $\theta_1=0$. All images under π_N of lines of this form intersect at $(1,0)$ and $(-1,0)$, so the resulting circles in the plane will still meet in angles that satisfy the same additive property as the lines in S^2.

Hence, to prove the case where both L_1 and L_2 intersect at $(\pm1,0,0)$, it suffices to consider the case where $L_1=E$ and $L_2:=L:=R_{z,\theta}E$ for some $0<\theta<\pi/2$. These two lines clearly meet at angle θ. We have $\pi_N(E)=E$, and from (b) we know that

$$\pi_N(L)=\left\{\left(\frac{\cos t}{1-\sin\theta\sin t},\frac{\cos\theta\sin t}{1-\sin\theta\sin t}\right) : t\in[0,2\pi)\right\}. \tag{7}$$

(We could instead use the constant-velocity parametrization found in (b), but then the values of t won't match up with the original parametrization of E)

Now, $\pi_N(L)$ meets E at $(\pm1,0)$ as usual, so we need to prove that these meetings have angle θ. Under the usual parametrization, the tangent vector to E at $(1,0)$ is $(0,1)^\mathsf{T}$. On the other hand, applying d/dt to 7 and evaluating at $t=0$ shows that the tangent vector to $\pi_N(L)$ at $(1,0)$ is $(\sin\theta,\cos\theta)^\mathsf{T}$. which certainly meets $(0,1)^\mathsf{T}$ at angle θ. The case at $(-1,0)$ is nearly identical. This proves the result for two lines which meet at $(\pm1,0,0)$ on the equator.

Back to the general case. We have assumed that L_1 contains $(\pm1,0,0)$, but L_2 meets E at some other pair of antipodal points. We'll now use ideas from the solution and Remark of Problem 3(e), where we make no distinction between $\mathbb{R}^2$ rotations in the xy-plane and $\mathbb{R}^3$ rotations about the z-axis. There is some angle φ such that $L_2' := R_{z,\varphi}^{-1}L_2$ contains $(\pm1,0,0)$, our usual standard form. Therefore, by the previous case, the angle θ between L_1 and L_2' is preserved by π_N.

From the result of Problem 3(e), we know that

$$\pi_N(L_2) = R_{z,\varphi}\big(\pi_N(L_2')\big)\,.$$

So it suffices to show that this rotation $R_{z,\varphi}$ of the circle $\pi_N(L_2')$ changes the angle θ made with $\pi_N(L_1)$ in exactly the same way as the rotation $R_{z,\varphi}$ of the line L_2' changes the angle made with L_1. Indeed, suppose $L_1 = R_{x,\psi_1}E$ and $L_2' = R_{z,\varphi}R_{x,\psi_2}E$. These are cut out by planes with normal vectors

$$\begin{aligned} R_{x,\psi_1}(0,0,1) &= (0,-\sin\psi_1,\cos\psi_1) \qquad \text{and} \\ R_{z,\varphi}R_{x,\psi_2}(0,0,1) &= (\sin\varphi\sin\psi_2,-\cos\varphi\sin\psi_2,\cos\psi_2), \end{aligned}$$

Taking the dot product of these, we see that the angle θ between L_1 and L_2' satisfies

$$\cos\theta = \cos\psi_1\cos\psi_2 + \cos\varphi\sin\psi_1\sin\psi_2. \tag{8}$$

(Notice that when $\varphi = 0$, we get $\cos\theta = \cos(\psi_1-\psi_2)$, as expected).

On the other hand, using the results of (b), $\pi_N(L_1)$ is the circle with center $(0,\tan\psi_1)$ and radius $r_1 = \sec\psi_1$. The data for $\pi_N(L_2')$ is similar, except that we should apply the rotation $R_{z,\varphi}$. This means that $\pi_N(L_2')$ has center

$$(-\sin\varphi\tan\psi_2,\cos\varphi\tan\psi_2)$$

and radius $r_2 = \sec\psi_2$.

Finally, we use a classical result from geometry: given two intersecting circles with radii r_1 and r_2, whose centers are separated by distance d, they intersect at an angle θ given by

$$\cos\theta = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}. \tag{9}$$

We know the centers of our circles $\pi_N(L_1)$ and $\pi_N(L_2')$, so we can find the distance d. Using all of this, and a lot of trigonometry, the right-hand side of 9 will match the right-hand side of 8. This was what we needed to show.

Problem 5. (20 pts) **Stereographic Projection for 1-sphere**. Let us consider the 1-sphere $S^1 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$ with its North Pole $N=(0,1)$, and $H = \{(x,y)\in\mathbb{R}^2 : y=0\}\subseteq\mathbb{R}^2$ the horizontal x-axis. Define the stereographic projection

$$\pi_N : S^1\setminus\{N\}\longrightarrow\mathbb{R},\quad \pi_N(P)=Q,$$

where $P = (x, y) \in S^1$, and $Q = u \in \mathbb{R}$ is defined as the unique intersection point $L_{P,N} \cap H$ distinct from N, and $L_{P,N} \subseteq \mathbb{R}^2$ is the unique line containing $P, N \in \mathbb{R}^2$.

(a) Find a formula of $\pi_N(P) = u$ in terms of $P = (x, y)$.

(b) Show that $(x, y) \in S^1 \setminus \{N\}$ has rational coordinates, i.e. $x, y \in \mathbb{Q}$, if and only if $\pi_N(x, y)$ is a rational number.

(c) Show that $\pi_N : (S^1 \setminus \{N\}) \cap \mathbb{Q}^2 \longrightarrow \mathbb{Q}$ is a bijection.

(d) Show that the inverse image $\pi_N^{-1}(u)$ of a rational point $u = m/n \in \mathbb{Q} \subseteq \mathbb{R}$, $m, n \in \mathbb{N}$, is given by
$$(x, y) = \left(\frac{2nm}{n^2 + m^2}, \frac{m^2 - n^2}{n^2 + m^2}\right).$$

(e) Let $m, n \in \mathbb{N}$ be two natural numbers, and define
$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$
Show that $a^2 + b^2 = c^2$.

(f) *Euclid's Theorem For Phytagorean Triples.* (Optional: Not graded) Let $a, b, c \in \mathbb{N}$ be a primitive Phytagorean triple, i.e. $a^2 + b^2 = c^2$ such that $\gcd(a, b, c) = 1$. Show that there exist $m, n \in \mathbb{N}$ such that
$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$

Solution.

(a) The line $L_{P,N}$ connecting $N = (0, 1)$ to $P = (a, b)$ is cut out by the equation
$$y - 1 = \frac{b - 1}{a} x,$$
which has x-intercept $a/(1 - b)$. Therefore,
$$u = \pi_N(x, y) = \frac{x}{1 - y}.$$
Notice that this resembles stereographic projection for the 2-sphere.

(b) The difference of two rational numbers is a rational number, and the quotient of two rational numbers is a rational number, so if $x, y \in \mathbb{Q}$ are rational, certainly $\pi_N(x, y) = x/(1 - y)$ is rational too.

Conversely, suppose $\pi_N(x, y)$ is rational, so $\pi_N(x, y) = m/n$ for some integers m, n with $m \neq 0$. Using the fact that $(x, y) \in S^1$ satisfies $x^2 + y^2 = 1$, we have
$$\frac{m}{n} = \pi_N(x, y) = \frac{x}{1 - y} = \pm \frac{\sqrt{1 - y^2}}{1 - y},$$
which has solution
$$y = \frac{m^2 - n^2}{m^2 + n^2}.$$

Therefore, y is rational. Since $x = (1-y)\pi_N(x,y) = (1-y)m/n$, we see that x is also rational.

(c) In (b) above, we showed that this map makes sense: that is, π_N actually does map $(S^1 \setminus \{N\}) \cap \mathbb{Q}^2$ (rational points in the domain) to $\mathbb{Q}$ (rational numbers in the target space). The full map $\pi_N : S^1 \setminus \{N\} \longrightarrow \mathbb{R}$ is already bijective (its inverse can be found as in Problem 2(b), or, if you like, this projection is really just a restriction of the higher-dimensional version to a particular great circle going through the North pole), so certainly our restriction is one-to-one.

Finally, we must show that it is onto, meaning that π_N hits every rational number. Indeed, we showed above in (b) that if $\pi_N(x,y) = m/n$, then the rational pair

$$(x,y) = \left(\frac{m}{n}\left(1 - \frac{m^2-n^2}{m^2+n^2}\right), \frac{m^2-n^2}{m^2+n^2}\right) = \left(\frac{2mn}{n^2+m^2}, \frac{m^2-n^2}{m^2+n^2}\right) \tag{10}$$

achieves the value m/n under the map π_N.

(d) We could find the full inverse function just like we did in Problem 2(b), but for rational numbers, we've already done the necessary work. We essentially solved for (x,y) in the proof of (b), and we wrote it explicitly in 10.

(e) We showed in (c) and (d) above that

$$(x,y) = \left(\frac{2mn}{n^2+m^2}, \frac{m^2-n^2}{m^2+n^2}\right) = \left(\frac{b}{c}, \frac{a}{c}\right)$$

is the solution to $\pi_N(x,y) = m/n$. In particular, it is a point on the circle S^1, so

$$1 = x^2 + y^2 = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2.$$

Multiplying through by c^2, we find $c^2 = b^2 + a^2$, as desired.

Alternatively, we can just compute:

$$a^2 + b^2 = (m^2-n^2)^2 + (2mn)^2 = (m^2+n^2)^2 = c^2.$$

Remark. Therefore, from elementary geometry, we have constructed an infinite set of Pythagorean triples! One triple for each pair of positive integers m, n.

(f) Since a, b, and c are pairwise coprime (meaning the gcd of any pair of them is 1), it cannot be true that both a and b are even. Since one of them is odd, by relabeling if necessary, we may assume that a is odd. If b were also odd, then

$$c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod 4,$$

(where we used that $1^2 \equiv 3^2 \equiv 1 \pmod 4$), which is a contradiction because nothing squares to 2 mod 4. Therefore, b is even. Finally, c must be odd (because two even numbers cannot be coprime).

Next, we form the positive integers

$$s = \frac{c+a}{2} \quad \text{and} \quad d = \frac{c-a}{2}.$$

(They are integers because c and a are both odd, so their sum and difference are both even. The latter is positive because $c > a$ is the hypotenuse of a right triangle.) These two numbers are also coprime, because any common divisor would divide $s + d = c$ and $s - d = a$, contradicting the assumption that a and c are coprime. Next,

$$b = \sqrt{c^2 - a^2} = \sqrt{(s+d)^2 - (s-d)^2} = 2\sqrt{sd}.$$

Since b is even, $\sqrt{sd}$ is an integer, so sd is a perfect square. Therefore, since s and d share no common factors, they must each be perfect squares, so write $s = m^2$ and $d = n^2$ for positive integers m and n. Then

$$m^2 - n^2 = s - d = a, \quad 2mn = 2\sqrt{sd} = b, \quad \text{and} \quad m^2 + n^2 = s + d = c$$

as desired.

Remark. This answers the question: "did part (e) actually construct *all* possible Pythagorean triples?" The answer is "almost". Given any Pythagorean triple (a, b, c), we can scale the corresponding triangle down by $\gcd(a, b, c)$. The resulting side lengths will still be integers, and they will form a primitive Pythagorean triple. Therefore, Euclid's formula in (e) produces all Pythagorean triples *up to rescaling*.