Notes on First Order Differential Equations for Math 22B,

based on Chapter 2 of Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems***.**

Method of Integrating Factors. (Section 2.1)

Assume we have a first order linear equation of the form

$$y' + p(t)y = g(t) \qquad (*)$$

Define the **integrating factor**

$$\mu(t) = \exp \int p(t)\, dt.$$

Multiplying both sides of equation $(*)$ by $\mu(t)$ then integrating with respect to t, we obtain the following general solution

$$y = \frac{1}{\mu(t)} \left[\int \mu(t) g(t)\, dt + c \right]$$

of the original equation, where c is an arbitrary constant. We may use an initial condition $y(t_0) = y_0$ to determine the constant c.

Separable Equations. (Section 2.2)

Assume we have a first order differential equation which we can write in the form

$$M(x) + N(y)(dy/dx) = 0.$$

This equation is said to be **separable**, and we may write

$$\int M(x)\, dx = -\int N(y)\, dy.$$

Given an an initial condition $y(x_0) = y_0$, the solution to this initial value problem is then given by

$$\int_{x_0}^{x} M(s)\, ds + \int_{y_0}^{y} N(s)\, ds = 0.$$

Exact Equations and Integrating Factors. (Section 2.6.)

[NOTE: Section 2.6 is not listed in the class syllabus. I'm including it here for those who are curious about exact equations.]

Theorem 2.6.1. Say we are given a first order differential equation of the form

$$M(x, y) + N(x, y)y' = 0 \qquad (**)$$

Assume the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, are continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. (Actually, it suffices to assume that the region is simply connected.) Then, there exists a function ψ satisfying

$$\psi_x(x, y) = M(x, y) \qquad \text{and} \qquad \psi_y(x, y) = N(x, y)$$

if and only if M and N satisfy

$$M_y(x, y) = N_x(x, y)$$

at each point of R. In this case, $(*)$ is said to be an **exact** differential equation in R, and $\psi(x, y)$ is an implicitly defined solution to $(*)$.

If we do not have $M_y(x, y) = N_x(x, y)$, we may be able to multiply the equation $(*)$ by an integrating factor $\mu(x, y)$ to convert it into an exact differential equation. Two particular cases of this are when μ depends only on x and when μ depends only on y.

(a) If $(M_y - N_x)/N$ is a function of x only, then solving the (linear and separable) differential equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N} \mu$$

gives an integrating factor $\mu(x)$, which we may use to convert $(*)$ into an exact differential equation.
(b) If $(N_x - M_y)/M$ is a function of y only, then solving the (linear and separable) differential equation

$$\frac{d\mu}{dx} = \frac{N_x - M_y}{M}\mu$$

gives an integrating factor $\mu(y)$, which we may use to convert $(*)$ into an exact differential equation.

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Differences Between Linear and Nonlinear Equations. (Section 2.4)

Theorem 2.4.1. Assume we are given the initial value problem

$$\begin{cases} y' + p(t)y = g(t), & (1) \\ y(t_0) = y_0, & (2) \end{cases}$$

where y_0 is an arbitrary prescribed initial value. If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation (1) for each t in I, and that also satisfies the initial condition (2).

Theorem 2.4.2. Assume we are given the initial value problem

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \qquad (*)$$

where y_0 is an arbitrary prescribed initial value. If the functions f and $\partial f/\partial y$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there is a unique solution $y = \phi(t)$ of the initial value problem $(*)$.

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Euler's Method. (Section 2.7)

Given an first order initial value problem

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0,$$

we may approximate the solution y_n of $y(t)$ at $t = t_n$ by iteratively computing

$$y_{k+1} = y_k + f(t_k, y_k) \cdot h,$$

for $k = 1, \ldots, n$, where $h = (t_n - t_0)/n$ and $t_k = t_{k-1} + h$.

Picard's Iteration Method (Section 2.8.)

Given an first order initial value problem

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0,$$

we may approximate a solution $y = \phi(t)$ by iteratively computing

$$\phi_k(t) = \int_0^t f[s, \phi_{k-1}(s)]\, ds,$$

then taking the limit as $k \to \infty$. We often choose the initial approximation $\phi_0(t)$ to be zero.