Notes on the Laplace Transform for Math 22B

based on Chapter 6 of Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems*.

Section 6.1: Definition of the Laplace Transform.

Definition. A function f is **piecewise continuous** on an interval $\alpha \le t \le \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

(i) f is continuous on each open subinterval $t_{i-1} < t < t_i$, and

(ii) f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

Theorem 6.1.1. Assume f is piecewise continuous for $t \ge a$.

(i) If $|f(t)| \le g(t)$ when $t \ge M$ for some positive constant M, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges.

(ii) If $f(t) \ge g(t) \ge 0$ for $t \ge M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

Theorem 6.1.2. Suppose that $f(t)$ is **piecewise continuous and of exponential order as** $t \to \infty$, i.e.

(i) $f(t)$ is piecewise continuous on the interval $0 \le t \le A$ for any positive A, and

(ii) $|f(t)| \le Ke^{at}$ when $t \ge M > 0$, where $K > 0$ and a are real constants.

Then, the **Laplace transform**

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

exists for $s > a$.

Fact. The Laplace transform is a linear operator. I.e., if f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively, then, for s greater than the maximum of a_1 and a_2,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 L\{f_2(t)\}.$$

Section 6.2: Solution of Initial Value Problems.

Theorem 6.2.1. Suppose that

(i) f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$, and

(ii) there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$.

Then, $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Corollary 6.2.2. Suppose that

(i) $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$, and

(ii) there exist constants K, a, and M such that, for $t \ge M$,

$$|f(t)| \le Ke^{at}, \qquad |f'(t)| \le Ke^{at}, \qquad \ldots, \qquad |f^{(n-1)}(t)| \le Ke^{at}$$

Then, $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Using the Laplace transform to solve a differential equation.

1. Use the above definition of $\mathcal{L}\{f(t)\}$ and the above results regarding $\mathcal{L}\{f^{(n)}(t)\}$ to transform an initial value problem for an unknown function f in the t-domain into an algebraic problem for F in the s-domain.

2. Solve this algebraic problem to find $F(s)$.

3. Recover the desired function f from its transform F by "inverting the transform," i.e.

The following table for inverse Laplace transforms is from page 321 of BDP. Note that the inverse Laplace transform is linear.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$