MATH 22AL
Computer LAB for Linear Algebra Dr. Daddel
Eigenvalues and Eigenvectors

Please save your MATLAB Session (diary)as "LAB9.text" and submit.

0.1 Eigenvalues and Eigenvectors

In this LAB we will cover the following topics regarding Eigenvalues and Eigenvectors.

- An Example
- Definitions
- How to find Eigenvalues
- How to find Eigenvectors
- Applications

How to do this LAB:

- 1. Format of this lab is different from previous labs, You start diary by typing " diary LAB9.text"

- 2. Then read the Lab, and when ever needed type the command in the MATLAB, continue reading and enter in MATLAB the commands appeared in the LAB.

- 3. In this LAB :

  - a.) When it says "let $v_1 = \mathbf{v_1} = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}$"
    it means Enter "$v1 = [\ 1 \ \ 6 \ \ -13\ ]'$" in MATLAB (note the transpose)

  - b.) When it says "Find $A\mathbf{v_3}$"
    it means Enter "$A * v3$" in MATLAB

  - c.) When you see "**Explain**", type the percentage character and enter your comments.

  - d.) When it says in the lab that "you should be able to prove this" That means it may be helpful in your Linear Algebra course be able to prove it. Do not provide proof here.

  - d.) When it says "Find " That means Type in MATLAB.

0.2 Example

Lets look at some examples of Eigenvalues and Eigenvectors, then have a quick look to an application:
Consider the square matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Enter A in MATLAB.

Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}$ Enter v_1 in MATLAB.

1. Find $A\mathbf{v_1}$ (Note: you need to type $A * v1$)
2. Find $A^2\mathbf{v_1}$

You will see that $Av_1 = 0$ and $A^2v_1 = 0$. So, v_1 is in the NULL space of A.

Now let Let $\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ Enter v_2 in MATLAB.

1. Find $A\mathbf{v_2}$
2. Find $A^2\mathbf{v_2}$
3. Find $A^3\mathbf{v_2}$

You will see that $Av_2 = -4v_2$ and $A^2v_2 = (-4)^2v_2$, So, multiplication by A stretches v_2 by factor of 4 and reverses the direction.

let Let $\mathbf{v_3} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$ Enter v_3 in MATLAB.

1. Find $A\mathbf{v_3}$
2. Find $A^2\mathbf{v_3}$
3. Find $A^3\mathbf{v_3}$

You will see that $Av_3 = 3v_3$ and $A^2v_3 = (3)^2v_3$, So, multiplication by A stretches v_3 by factor of 3 and preserves the direction.

Now let's choose a random vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ Enter u in MATLAB.

1. Find $A\mathbf{u}$
2. Find $A^2\mathbf{u}$
3. Find $A^3\mathbf{u}$

You will see that multiplication by A changes $\mathbf{u}$ to new vector $A\mathbf{u}$ which is completely different vector and is not in the direction of $\mathbf{u}$ or its opposite. For the matrix A above the vectors $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ are special vectors, multiplication by A will keep the direction of the these vectors, it might stretch, compress or collapse them in the same direction or opposite direction.

0.3 Definitions

This section is just for your reading, all is covered in your 22A class.
1. Let A be an $n \times n$ matrix, the number λ is called an **eigenvalue** of A if there exists a nonzero vector $x$such that $Ax = \lambda x$.

2. The vector $x \neq 0$ is called an **eigenvector** corresponding to λ if $Ax = \lambda x$.
3. The equation $Ax = \lambda x$ is equivalent to $(A - \lambda I)x = 0$.
4. All of the following are equivalent:

- λ is an eigenvalue of A.
- $(A - \lambda I)x = 0$ has a nontrivial solution.
- $A - \lambda I$ is singular.
- $det(A - \lambda I) = 0$.

All non-zero solutions of $(A - \lambda I)x = 0$ are the eigenvectors for λ
These vectors together with the 0 vector form a subspace which is called **the eigenspace corresponding to eigenvalue λ.**
The expression $det(A - \lambda I)$ is a polynomial of degree n of λ, which is called **the characteristic polynomial of** A.
Note that the eigenvalues are the roots of **the characteristic equation** $det(A - \lambda I) = 0$.

0.4 An Application

Now consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

enter A in MATLAB

with the three eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

Form Matrix P whose columns are these three vectors.

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 6 & 2 & 3 \\ -13 & 1 & -2 \end{bmatrix}$$

Use MATLAB to find det(P).

means enter A, v_1, v_2, v_3 and P in MATLAB and then enter " det(P)"

You should get 84. So P is invertible.

Use MATLAB to find inverse of P. means type "inv(P)"

Now find $D = P^{-1}AP$. By typing "D=inv(P)*A*P".

Notice that we have a diagonal matrix with "eigenvalues " of A on the diagonal entry of D. You can solve $D = P^{-1}AP$ for A as
$A = PDP^{-1}$.

This part is just for your reading, all is covered in your 22A class.

Now you need a pen and paper to convince yourself that $A^2 = A * A = (PDP^{-1}) * (PDP^{-1}) = (PD^2P^{-1})$ and prove that

$$A^{200} = A*A*A*\ldots*A*A = (PDP^{-1})*(PDP^{-1})*\ldots*(PDP^{-1})*(PDP^{-1}) = (PD^{200}P^{-1})$$

Now compute D^2, D^3 and D^4 this will show that D^n is a diagonal matrix whose diagonal entries are the diagonal entries of D raised to the n^{th} power.
So for any given vector u it is easier to find $A^{200}u$ as $A^{200}u = PD^{200}P^{-1}u$
This is only an example of many applications of eigenvalues and eigenvectors.

0.5 How to find Eigenvalues and Eigenvectors using MATLAB

Consider a matrix A

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

enter A in MATLAB
Type: d = eig(A)
This will show you a vector of the eigenvalues of matrix A. But might not look pretty!!!
type : format rat
type : d = eig(A)

It should give you eigenvlaues in rational form, like this:
-4
3
1/3850048286914616

The last one is the zero.
Type : format long
Type: d = eig(A)
you will see
-3.999999999999994
3.000000000000001
0.000000000000000

which basically represents -4, 3 and 0. This is true also when you use matlab to find eigenvectors:
type: [V,D] = eig(A)
this will produces two matrices:
A diagonal matrix D. On the main diagonal of D are eigenvalues of A , and a matrix V whose columns are eigenvectors of A corresponding to the eigenvalues in the diagonal of D.

You may see:

$$V = \begin{bmatrix} 0.408248290463863 & -0.485071250072666 & -0.069673301429162 \\ -0.816496580927726 & -0.727606875108999 & -0.418039808574970 \\ -0.408248290463863 & 0.485071250072666 & 0.905752918579103 \end{bmatrix}$$

The first column is a multiple of $\mathbf{v_1} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Do this in MATLAB:
If you multiply the first column by -5 and divide by 2 and round it up, you will get approximately $\mathbf{v_1}$.

Now Find AV and VD for matrix A, they must be equal, Are they?

0.5.1 Does row operation preserve eigenvalues?

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Let B obtained by interchanging the first and second row of A. To obtain B, define the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Enter A, B and P in MATLAB
Find $B = PA$ (recall to type P*A in MATLAB)
Type: eig(B) to find eigenvalues of B

Observe that eigenvalues changed because of the row operation done on the matrix.
So row operations will change the eigenvalues of a matrix

0.5.2 What is the effect of adding a multiple of I(identity matrix) to A?

let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Enter A in MATLAB

Type: eig(A)
Type: A1=A-3*eye(3)
Type: eig(A1)
Type: A2=A+4*eye(3)
Type: eig(A2)
Type: A3=A-5*eye(3)
Type: eig(A3)
Type: A4=A-10*eye(3)

Type: eig(A4)

answer this question If 3 is an eigenvalue of a matrix M what is an eigenvalue of $M - 6I$?
You should be able to generalize and prove this observation as : If k is an eigenvalue of a matrix M what is an eigenvalue of $M - sI$?

0.5.3 Are eigenvalues of A and A^{-1} are related?

Let

$$C = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

Enter C in MATLAB and
Find det(C) to verify that A is invertible.
Find inverse of C by typing inv(C).
Find eigenvalues of both C and C^{-1}.

Explain what you see and generalize your observation.
You should be able to prove it. Not here , in your 22A class.
Compare eigenvectors of C and C^{-1}, generalize.
Explain your generalization.

0.5.4 Are eigenvalues of A and A^n are related?

Let

$$C = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

Enter C in MATLAB
Find eigenvalues of both C and C^2.
Find eigenvalues of C^3, C^5.
Explain what you see and generalize your observation.
You should be able to prove it. Not here , in your 22A class.
Experiment with C, C^2, C^3 and C^5.

Answer Is there any relation between eigenvectors of C and eigenvectors of C^n?

0.5.5 Two important properties of eigenvalues

Recall that trace of a square matrix is the sum of the entries in main diagonal.
Let

$$C = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Enter A and C in MATLAB
Find trace of C and A by typing : trace(A) and trace(C)
Find eigenvalues of A and C.
Explain Do you see any relation between eigenvalues and trace of of a matrix?
Find det(A) and det(C)
Explain Do you see any relation between eigenvalues and determinant of a matrix?