Remarks on Monday July 8 in Bern, Switzerland

## 1. Rebuttal

I'm not sure why this happened. I'm not retiring, I'm not sick, my 66th birthday was five months ago and this would only be a landmark birthday if we all had an extra finger.

I profoundly thank you all for coming to this wonderful event.

My first bonus set of thanks are to Jesus and Vicki for accomplishing the incredible task of organizing and publicizing this event and getting such a gratifying turnout.

I also want to thank the speakers: Vicki, Bernd, Frank, Pablo, Greg, Salma, Jesus and Robert for their many kind and generous remarks and for taking the time to prepare them today.

My brother Robert, sister-in-law Pam and nephew Pete interrupted their European vacation to come here and mingle with the mathematicians. I'm very grateful for the effort they put into finding and annotating a lot of old family pictures. My niece Emma is a grad student and couldn't come, but you can find her latest paper on the arXiv.

I thank you all again for taking the time to be here and leave you with an incomplete idea that maybe some of you will find interesting.

The course of my future personal and professional life was set in the first week of November in 1976. That week, at the suggestion of Raphael M. Robinson, I wrote to his colleague T. Y. Lam, requesting copies of his preprints on Hilbert's 17th Problem. When I saw them, I realized that this was a much richer and more interesting subject than my recent PhD thesis. Any mathematical work of mine which you heard about today is a direct consequence of the inspiration these two papers have continued to give me over 40 plus years.

The same week, I invited a math graduate student named Robin Sahner to dinner and a play, neither of which she really liked. Nonetheless, we have been together ever since. I am also grateful that she overcame her reluctance to fly in order to be here today and hear what you all had to say. In addition to making my brilliant shirts, Robin's top two publications do better on Google Scholar than mine; her book on computer performance with her PhD advisor has more than 800 citations. Robin is a magnificent human being and I am grateful beyond words for our decades together.

## 2. The talk that nobody could understand

I gave talks on this subject at Oberwolfach, Caltech and Urbana in the mid 80s and nobody could figure out what I was talking about, probably because the geometric interpretation of exponents in a polynomial wasn't so familiar. I will present this in two variables, though there are obvious extensions to any number of variables. The name I gave to the topic was "deeper arithmetic-geometric inequalities". My naming skills weren't so good back then.

Suppose $k$ points are given in the plane: $(a_i, b_i)$ and we consider a polynomial supported on these points:

$$p(x,y) = \sum c_i x^{a_i} y^{b_i}.$$

We do not assume that $a_i, b_i$ are integers and so we restrict to $x, y > 0$. Suppose we require that $p$ vanish to first order at $(1,1)$, and that the points are such that there is a unique (projective) solution $p$ and that all points $(a_i, b_i)$ are necessary, so no $c_i = 0$. The equations are $\sum c_i = \sum c_i a_i = \sum c_i b_i = 0$. We want to know the conditions on the points so that (up to sign) $p(x,y) \geq 0$ on $\mathbb{R}^2_+$.

It is neither hard nor interesting to show that these conditions are satisfied either if $k = 3$ and the points are on a line, or if $k = 4$ and no three points are on a line. In the first case, Descartes' Rule of Signs says that $p \geq 0$ and the resulting $p$ is what I used to call an agiform (a form derived from monomial substitution into the Arithmetic-Geometric Inequality.) In the second case, there are in fact two cases: either each point is a vertex of the convex hull, or there is a triangle with a single point inside. In the first case, $p$ is not psd, because the $c_i$'s take both signs. In the second case, $p$ is an agiform and so is psd.

The same thing holds for general $n$: $p \geq 0$ is equivalent to the geometric shape of the given points being one point inside a simplex, and the form is an agiform.

So the question I asked, was, what happens in the "next" case: when you replace "vanish to first order" with "vanish to third order"? (Or of course, more generally to $(2m-1)$-st order.)

Again, in one variable, Descartes' Rule of Signs says that every form you get is psd. In the plane, there are 10 equations. Here is one very special example I wish I had thought of 35 years ago. Let the 11 points be the 10 integer points $(i, j) : 0 \leq i, j, i + j \leq 3$ plus one other point $(a, b)$,. Then there is exactly one choice of $c_{ij}$ so that

$$p(x,y) = x^a y^b - \sum_{i=0}^{3} \sum_{j=0}^{3-i} c_{ij} x^i y^j$$

vanishes to third order at (1,1), and that is when the sum above is the Taylor approximation to $x^a y^b$ at $(1, 1)$. In this case, the difference $p(x, y)$ is the *error* to the Taylor approximation, which can be expressed exactly as a fourth derivative of $x^a y^b$ in some direction at some point. (There are restrictions on $(a, b)$ based on $c_{ij} \neq 0$.) In this way, I can show that if $(a, b)$ lies in the open triangle with vertices $(0, 0), (1, 0), (0, 1)$, then the resulting form is psd. I can handle a few other choices of $(a, b)$ in this special case.

Big questions:

1. What shapes correspond to $p \geq 0$?

2. In the line, Descartes' Rule of Signs insures that the coefficients of the monomials alternate in sign. What is the two-dimensional version of this? Must there be concentric layers of positive and negative coefficients?

Recreated from memory and notes on July 26, 2019