Champaign Problems

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Bruce Fest,
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Pre-Show Ad

Work started on a book-length project: Representations of Positive Polynomials

The idea: Gather together and organize results on "certificates of positivity" with some history. Put in enough background material and details so that it is useful for graduate students and non-experts.

Looking for suggestions, results that should be included (especially recent results), anyone who wants to read and comment on parts that are written...

Timeline

Hilbert, 1888

Bernstein, 1915

Hamburger, 1920

Pólya, 1928

Haviland, 1936

Motzkin, 1967

Krivine, 1964

Berg et. al., 1979

Stengle, 1974

Handelman, 1988

Schmüdgen, 1991

CTC Wall, 1991

Choi, Lam, Rez, 1995

Powers, Reznick, 2001

Scheiderer, 2000

Schweighofer, 2002

Kuhlmann, Marshall, 2002

PRSS, 2004

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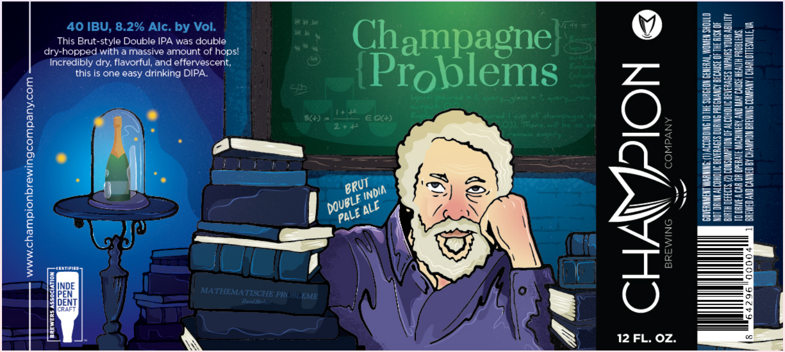

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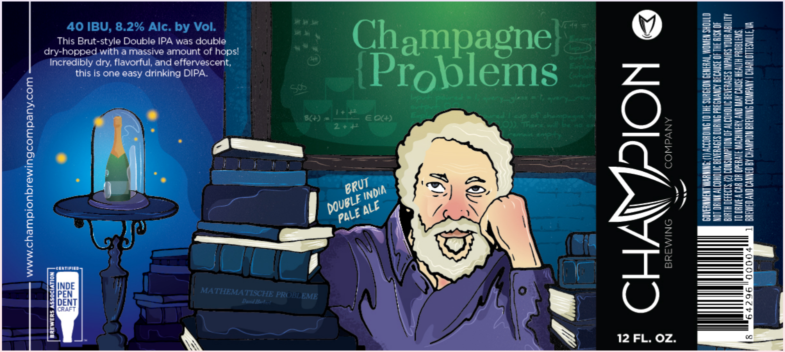

and not some wimpy lager, an 8.2% double IPA!

Higher Level Orderings on a Field

Hilbert's 17th Problem: If $f \in \mathbb{R}(X)$ is psd, then $f \in \sum \mathbb{R}(X)^2$. Solved by E. Artin in 1927, using the newly developed theory of ordered fields of Artin and Schreier.

For a field F, $P \subseteq F$ is (the positive cone of) an **order** on F if $P \cdot P \subseteq P$, $P + P \subseteq P$, $P \cap -P = \{0\}$, $P \cup -P = F$. Equivalently, $P \subseteq F$ is an order if $\dot{P}$ is a subgroup of $\dot{F}$ of index 2 that is additively closed.

In the 1970's, Eberhard Becker developed a generalization: An **ordering of higher level** on F is a subset $P \subseteq F$ such that $\dot{P}$ is an additively closed subgroup of $\dot{F}$ for which $\dot{F}/\dot{P}$ is finite cyclic. $\dot{P}$ additively closed implies $-1 \notin P$, hence $\dot{F}/\dot{P}$ has order $2n$. The **level** of P is n. Orders in F are level 1 orderings.

Artin-Schreier Theory

A field F is (formally) **real** if $-1 \notin \sum F^2$.

Artin-Schreier Theorem

F is real iff F admits an order. Furthermore, if F is real then $\sum F^2 = \cap P$, the intersection of all orders in F.

Artin's solution to Hilbert's 17th Problem: Showed that $F \subseteq \mathbb{R}$ a subfield with exactly one order, then for any $f \in F(X)$ such that $f(x) \geq 0$ at every point at which it is defined, then f is in every order of $F(X)$. Used Sturm's Theorem for counting the number of real roots of a polynomial.

In 1978, Becker proved a higher level version:

Higher Level Artin-Schreier Theorem

The following are equivalent for a field F and any $n \in \mathbb{N}$.

1. *F has an ordering of level n.*
2. $-1 \notin \sum F^{2n}$.
3. *F is real.*

If F is real, then $\sum F^{2n} = \cap P$, where the intersection is over all orderings of level dividing n.

The Champagne Problem

Becker used valuation theory + higher level Artin-Scheirer to show that the following is true in a real field F for any $n \in \mathbb{N}$:

$$\left\{ \frac{r(s+q)}{t+q} \mid r, s, t \in \mathbb{Q}^+, q \in \sum F^2 \right\} \subseteq \sum F^{2n}$$

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As an example,

$$B(t) := \frac{1+t^2}{2+t^2} \in \sum \mathbb{Q}(t)^{2n}$$

for all $n \in \mathbb{N}$. Becker's proof is non-constructive, and he proposed the following problem: Find an explicit formula for writing $B(t)$ in $\sum \mathbb{Q}(t)^{2n}$. He offered a case of champagne for anybody who could solve this. Thus was born the Champagne Problem.

Becker also showed that for any real field F and $n \in \mathbb{N}$,

$$(\sum F^2)^n \subseteq \sum F^{2n}.$$

This is highly non-obvious!

More generally, Becker showed that there are identities

$$(x_1^{2n} + \cdots + x_k^{2n})^m = f_1^{2nm} + \cdots + f_r^{2nm},$$

where $f_i \in \mathbb{Q}(x_1, \ldots, x_k)$. When $n = 1$, these are the Hilbert Identities.

Back to the Champagne Problem. What did Bruce do?

We have

$$B(t) = \frac{1+t^2}{2+t^2} = \frac{(1+t^2)(2+t^2)^{2n-1}}{(2+t^2)^{2n}},$$

thus if we can write $(1+t^2)(2+t^2)^{n-1}$ and $(2+t^2)^n$ as sums of $2n$-th powers of elements of $\mathbb{Q}(X)$, their product is such a sum and can be divided by $(2+t^2)^{2n}$ yielding a formula in $\sum \mathbb{Q}(X)^{2n}$.

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In Bruce's paper *Uniform Denominators in Hilbert's 17th Problem*, Math. Zeit. 1995, using the above and the Hilbert Identities, after several pages of calculations, the following formula appears

$$B(t) := \frac{2^{4n-2}}{n(n+2)^2\binom{2n}{n}^2}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\lambda_j\left(\frac{L_i(\sqrt{2},t)L_j(\sqrt{2},t)}{2+t^2}\right)^{2n},$$

where $\lambda_j = 3n - (n+1)\cos(2j\pi/(n+2))$ and
$L_i(x,y) = (\cos(2j\pi/(n+2))x + \sin(2j\pi/(n+2))y)$.

This is an explicit formula for writing $B(t)$ in $\sum \mathbb{R}(t)^{2n}$, i.e., a solution to the Champagne Problem over $\mathbb{R}$. Unfortunately, this method does not lead to a solution over $\mathbb{Q}$. However, it was close enough that Becker awarded Bruce a bottle of champagne.

At the 1994 Joint Math Meeting, Bruce talked on this work at a special session and was given his champagne (by proxy).

THANK YOU Bruce for many years of collaboration, support, friendship, and laughs

Thanks to the audience for listening.